J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 1

Chapter - 1

1.1 Partial differential equations and their formation

Def. Differential Equation : An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. In connection with this basic definition, we do not include the differential equations which are derivative identities, for example,
$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}; \; \frac{d}{dx}\left(u.v\right) = u\frac{dv}{dx} + v\frac{du}{dx} \; \text{ etc.}$$

Def. Partial differential equation (P.D.E.) : A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

When we consider the case of two independent variables and one dependent variable, we usually take x and y as independent variables and z as dependent variable.

For example

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \qquad ......(1)$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right) \qquad ......(2)$$

$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^2 \qquad ......(3)$$

$$y\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = z\frac{\partial z}{\partial y} \qquad ......(4)$$

Def. Order of a Partial Differential Equation :

The order of the highest partial derivative occurring in the partial differential equation is called the order of a partial differential equation.

The equation (1) and (4) given above are of the first order, (3) is of second order and (2) is of the third order.

Def. Degree of a partial differential equation : The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized i.e., made free from radicals and fractions as far as derivatives are concerned. The equations (1), (2) given above are of first degree while equations (4) is of second degree.

Classification of Ist order PDE :

1. Linear PDE : A first order PDE is said to be linear if it is linear in p, q and z and of the form
$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

e.g., $p + q = z + xy$, $xp + yq = x$

(i) If $S(x, y) = 0$; then it is called homogeneous linear PDE.

(ii) If $S(x, y) \neq 0$; then it is called non-homogeneous linear PDE.

2. Semi-linear PDE : A first order partial differential equation is said to be semi-linear if it is linear in p and q but not necessarily in z and of the form $P(x, y) p + Q(x, y)q = R(x, y, z)$

e.g., (i) $p + q = xyz^2$, (ii) $xy^2 p + q = \dfrac{xy}{z^2}$

3. Quasi linear PDE : A first order PDE is said to be Quasi linear if it is linear in p and q and of the form

$$P(x, y, z) p + Q(x, y, z)q = R(x, y, z)$$

e.g., (i) $xyp + zq = xy$ (ii) $xz^2 p + q = xyz$

4. Non linear PDE : A first order partial differential equation is said to be non linear if it does not come under any one of the above type.

e.g., (i) $p^2 + q = 1$, $p^3 + q^3 = x$

Note : Linear $\Rightarrow$ Semi linear $\Rightarrow$ Quasi linear

Fomation of a partial differential equation by the elimination of arbitrary constants :

(Case of two arbitrary constants and two independent variables)

Consider an equation $F(x, y, z, a, b) = 0$ ......(1)

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y.

Differentiating (1) with respect to x and y partially, we get

$$\dfrac{\partial F}{\partial x} \cdot \dfrac{\partial x}{\partial x} + \dfrac{\partial F}{\partial y} \cdot \dfrac{\partial y}{\partial x} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial z}{\partial x} = 0 \quad \text{or} \quad \dfrac{\partial F}{\partial x} + p\left(\dfrac{\partial F}{\partial z}\right) = 0 \qquad ......(2)$$

and $\quad \dfrac{\partial F}{\partial x} \cdot \dfrac{\partial x}{\partial y} + \dfrac{\partial F}{\partial y} \cdot \dfrac{\partial y}{\partial y} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial z}{\partial y} = 0 \quad \text{or} \quad \dfrac{\partial F}{\partial y} + q\left(\dfrac{\partial F}{\partial z}\right) = 0 \qquad ......(3)$

Eliminating arbitrary constants a and b from the equations (1), (2) and (3) we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0 \qquad ......(4)$$

which is required partial differential equation of the first order.

Various situations in the formation of partial differential equation :

Situation (i) : When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually give rise to more than one partial differential equation of order one.

Situation (ii) : When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants give rise to a unique partial differential equation

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

of order one. Sometimes, it may not always be possible to eliminate the arbitrary constants from the given equation and its first order partial derivatives. Then we find the second order partial derivatives and eliminate the arbitrary constants. However, this higher order partial differential equation is not unique.

Situation (iii) : When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants give rise to a partial differential equation of order usually greater than one. However this higher order partial differential equation is not unique.

Example : Eliminate arbitrary constants a and b from $z = (x-a)^2 + (y-b)^2$ to form the partial differential equation.

Solution : Given function is $\qquad z = (x-a)^2 + (y-b)^2$ $\qquad$ ......(1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = 2(x-a) \qquad\qquad ......(2)$$

and

$$\frac{\partial z}{\partial y} = 2(y-b) \qquad\qquad ......(3)$$

To eliminate the arbitrary constant a and b from (2) and (3), squaring and adding (2) and (3), we get

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x-a)^2 + 4(y-b)^2 = 4\left[(x-a)^2 + (y-b)^2\right]$$

or $\qquad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$ $\qquad\qquad$ [By (1)]

which is the required partial differential equation.

Exercise 1.1

Eliminate the arbitrary constants from the following equations and form the corresponding partial differential equation

1. $z = ax^3 + by^3$
2. $4z = \left(ax + \dfrac{y}{a} + b\right)^2$
3. $z^2 = ax^3 + by^3 + ab$

4. $z = ax^2 + bxy + cy^2$
5. $z = (x-a)^2 + (y-b)^2$
6. $z = ax + a^2y^2 + b$

7. $z = (x+a)(y+b)$
8. $z = (x^2+a)(y^2+b)$
9. $z = a(x+y) + b(x-y) + abt + c$

10. $z = A\, e^{-p^2 t} \cos p\, x$

Answers

1. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z$
                                2. $z = \left(\dfrac{\partial z}{\partial x}\right)\left(\dfrac{\partial z}{\partial y}\right)$

3. $9x^2 y^2 z^2 = 6x^3 y^2 \left(\dfrac{\partial z}{\partial x}\right) + 6x^2 y^3 z \left(\dfrac{\partial z}{\partial y}\right) + 4z^2 \left(\dfrac{\partial z}{\partial x}\right)\left(\dfrac{\partial z}{\partial y}\right)$

4. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$
        5. $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = 4z$
    6. $\dfrac{\partial z}{\partial y} = 2y\left(\dfrac{\partial z}{\partial x}\right)^2$

7. $z = \left(\dfrac{\partial z}{\partial x}\right)\left(\dfrac{\partial z}{\partial y}\right)$
        8. $4xyz = \left(\dfrac{\partial z}{\partial x}\right)\left(\dfrac{\partial z}{\partial y}\right)$
    9. $\left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2 = 4\dfrac{\partial z}{\partial t}$

10. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial t}$

1.2 Formation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

Example : Form a partial differential equation by eliminating the arbitrary function f from the equation $x + y + z = f\left(x^2 + y^2 + z^2\right)$.

Solution : Given that
$$x + y + z = f\left(x^2 + y^2 + z^2\right) \qquad \qquad ......(1)$$

Differentiating (1) partially w.r.t. 'x' and 'y' respectively, we get

$$1 + p = f'\left(x^2 + y^2 + z^2\right)\left(2x + 2zp\right) \qquad \qquad ......(2)$$

and
$$1 + q = f'\left(x^2 + y^2 + z^2\right)\left(2y + 2zq\right) \qquad \qquad ......(3)$$

To eliminate $f'\left(x^2 + y^2 + z^2\right)$ from (2) and (3), we divide (2) by (3), to get

$$\frac{(1 + p)}{(1 + q)} = \frac{(2x + 2zp)}{(2y + 2zq)}$$

or
$$(1 + p)\,(y + zq) = (1 + q)(x + zp)$$

or
$$(y - z)p + (z - x)q = x - y$$

which is the required partial differential equation.

Exercise 1.2

Form a partial differential equation by eliminating the arbitrary functions from the following equations

1. $f\left(x + y + z,\ x^2 + y^2 - z^2\right) = 0$
        2. $z = f\left(x - ay\right) + g\left(x + ay\right)$

3. $lx + my + nz = \phi\left(x^2 + y^2 + z^2\right)$
        4. $z = e^{ax + by}\ f\left(ax - by\right)$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

5. $z = f(x+iy) + g(x-iy)$

6. (i) $z = f(x^2 - y^2)$ (ii) $z = f(x^2 + y^2)$

7. $z = x f(x+y) + g(x+y)$

8. $xyz = f(x+y+z)$

9. $z = y f_1(x) + x f_2(y)$

10. $f(x^2 + y^2 + z^2, \ z^2 - 2xy) = 0$

11. $z = e^y \ f(x+y)$

12. $z = y^2 + 2s\left(\dfrac{1}{x} + \log y\right)$

13. $z = f(xy) + \phi\left(\dfrac{x}{y}\right)$

14. $z = f\left(\dfrac{xy}{z}\right)$

15. $z = e^{ax-by} \ f(ax+by)$

Answers

1. $(y+z)p - (x+z)q = x - y$

2. $\dfrac{\partial^2 z}{\partial y^2} = a^2 \ \dfrac{\partial^2 z}{\partial x^2}$

3. $(ny - mz)\dfrac{\partial z}{\partial x} + (lz - nx)\dfrac{\partial z}{\partial y} = mx - ly$

4. $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$

5. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

6. (i) $yp + xq = 0$ (ii) $yp - xq = 0$

7. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$

8. $px(y-z) + qy(z-x) = z(x-y)$

9. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = xy \ \dfrac{\partial^2 z}{\partial x \partial y} + z$

10. $(p-q)z = y - x$

11. $q = z + p$

12. $x^2 p + yq = 2y^2$

13. $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} - y \dfrac{\partial z}{\partial y} = 0$

14. $px - qy = 0$

15. $2abz = b\dfrac{\partial z}{\partial x} - a\dfrac{\partial z}{\partial y}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

Chapter - 2

2.1 First order linear partial differential equations

Solution of a linear partial differential equation by direct integration : Some simple type of linear partial differential equation can be solved by direct integration as illustrated in the following example.

Example : Solve the following partial differential equations by direct integration :

(i) $\dfrac{\partial^2 z}{\partial x \partial y} + 7xy - \cos(2x+3y) = 0$ 　　　　(ii) $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(x+4y)$

Solution : (i) Given equation can be written as $\dfrac{\partial^2 z}{\partial x \partial y} = \cos(2x+3y) - 7xy$

Integrating both sides w.r.t. x treating y as constant, we get

$$\frac{\partial z}{\partial y} = \frac{\sin(2x+3y)}{2} - \frac{7}{2}x^2\,y + \phi(y) \qquad \text{where } \phi \text{ is a arbitrary function}$$

Again integrating both sides w.r.t. y treating x as constant, we get

$$z = -\frac{\cos(2x+3y)}{6} - \frac{7}{4}x^2\,y^2 + \int \phi(y)\,dy + \psi(x) \quad \text{where } \psi \text{ is a arbitrary function.}$$

(ii) Given equation is 　　　　$\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(x+4y)$

Integrating both sides w.r.t. x treating y as constant, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \sin(x+4y) + \phi(y) \quad \text{where } \phi \text{ is an arbitrary function.}$$

Again, integrating w.r.t. x treating y as constant, we get

$$\frac{\partial z}{\partial y} = -\cos(x+4y) + x\,\phi(y) + \psi(y)$$

Now integrating w.r.t. y treating x as constant, we get

$$z = -\frac{\sin(x+4y)}{4} + x\int \phi(y)\,dy + \int \psi(y)\,dy + \xi(x)$$

where ϕ, ψ and ξ are arbitrary functions.

Exercise 2.1

Solve the following partial differential equations by direct integration :

1. $\dfrac{\partial z}{\partial x} = x+y$ 　　　　2. $\dfrac{\partial z}{\partial y} = e^{x-y}$ 　　　3. $\dfrac{\partial^2 z}{\partial x^2} = xy$ 　　　4. $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$

5. $\dfrac{\partial^2 z}{\partial x \partial y} = e^{-y} \cos x$ 6. $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(3x+4y)$ 7. $\dfrac{\partial^3 z}{\partial x \partial y^2} - 18xy = e^x$

Answers

In all the answers below ϕ, ψ and ξ being arbitrary functions.

1. $z = \dfrac{x^2}{2} + xy + \phi(y)$ 2. $z = -e^{x-y} + \phi(x)$

3. $z = \dfrac{x^3}{6} y + x\phi(y) + \psi(y)$ 4. $z = -\dfrac{\sin(xy)}{x^2} + y\phi(x) + \psi(x)$

5. $z = -e^{-y} \sin x + \int \phi(y)\,dy + \psi(x)$

6. $z = -\dfrac{1}{36} \sin(3x+4y) + x\int \phi(y)\,dy + \int \psi(y)\,dy + \xi(x)$

7. $z = \dfrac{y^2}{2} e^x + \dfrac{3}{2} x^2 y^3 + \int \left\{ \int \phi(y)\,dy \right\} dy + y\psi(x) + \xi(x)$

2.2 Lagrange's Method based on type I

Lagrange's Equation : A partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y and z is known as Lagrange equation.

For example, $xyp + yzq = zx$ is a Lagrange equation.

Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are functions of x, y, z :

Theorem : The general solution of Lagrange equation

$$Pp + Qq = R \qquad\qquad ......(1)$$

is

$$\phi(u, v) = 0 \qquad\qquad ......(2)$$

where ϕ is an arbitrary function and $\quad u(x, y, z) = c_1 \quad$ and $\quad v(x, y, z) = c_2 \qquad ......(3)$

are two independent solution of $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ $\qquad\qquad ......(4)$

Here c_1 and c_2 are arbitrary constants and at least one of u, v must contain z.

Remark : Equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ are called Lagrange's auxiliary (or subsidiary) equation for

$Pp + Qq = R$.

Working Rule for solving $Pp + Qq = R$ by Lagrange's method :

(i) Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R \qquad\qquad ......(1)$$

(ii) Write down Lagrange's auxiliary equations for (1) namely,

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \qquad ......(2)$$

(iii) Solve (2) and let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2)

(iv) The general solution of (1) is then written in one of the following three equivalent forms :

$$\phi(u,v) = 0, \qquad u = \phi(v) \qquad \text{or} \qquad v = \phi(u), \phi \text{ being an arbitrary function}$$

Type I : Suppose that one of the variables is either absent or cancels out from any two fractions of

equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$. Then an integral can be obtained by the usual methods.

Example : Solve $a(p + q) = 1$.

Solution : The given equation is $\qquad\qquad ap + aq = 1 \qquad\qquad ......(1)$

which is of the form $Pp + Qq = R$.

The Lagrange's auxiliary equation are $\qquad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\text{or} \qquad\qquad \frac{dx}{a} = \frac{dy}{a} = \frac{dz}{1} \qquad\qquad ......(2)$$

Taking the first two fractions of (2), $\dfrac{dx}{a} = \dfrac{dy}{a} \qquad \Rightarrow \qquad dx - dy = 0 \qquad ....(3)$

Integrating it we get, $\qquad\qquad x - y = c_1 \qquad\qquad\qquad ......(4)$

where c_1 is an arbitrary constant.

Taking the last two members of (2), $\dfrac{dy}{a} = \dfrac{dz}{1} \qquad$ or $\qquad dy - adz = 0$

Integrating it, we have $\qquad\qquad y - az = c_2$, c_2 being an arbitrary constant.

Therefore the required solution is given by

$$\phi(x - y , y - az) = 0, \phi \text{ being an arbitrary function.}$$

Exercise 2.2

Solve the following partial differential equations :

1. $pz = x$

2. $\dfrac{y^2 z}{x} p + xzq = y^2$

3. $p \tan x + q \tan y = \tan z$

4. $(-a + x)p + (-b + y)q = -c + z$

5. $xp + yq = z$

6. $p + q = \sin x$

7. $y^2 p - xyq = x(z - 2y)$

8. $yzp + 2xq = xy$

9. $x^2 p + y^2 q + z^2 = 0$

10. $yzp + zxq = xy$

Answers

In all the answers below , ϕ being arbitrary function.

1. $\phi(y, x^2 - z^2) = 0$

2. $\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$

3. $\dfrac{\sin x}{\sin y} = \phi\left(\dfrac{\sin y}{\sin z}\right)$

4. $\phi\left\{ \dfrac{(x-a)}{(y-b)} , \dfrac{(y-b)}{(z-c)} \right\} = 0$

5. $\phi\left(\dfrac{x}{z} , \dfrac{y}{z}\right) = 0$

6. $\phi(x - y, z + \cos x) = 0$

7. $\phi\left(x^2 + y^2, zy - y^2\right) = 0$

8. $\phi\left(x^2 - z^2, y^2 - 4z\right) = 0$

9. $\phi\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{1}{y} + \dfrac{1}{z}\right) = 0$

10. $f(x^2 - y^2, y^2 - z^2) = 0$

2.3 Lagrange's Method based on type II

Type II : In this type one solution is obtained by the method of type (I) and then this solution is used to find another solution as explained in the following examples.

Example : Solve $p + 3q = 5z + \tan(y - 3x)$.

Solution : Given equation is $p + 3q = 5z + \tan(y - 3x)$ ......(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$$ ......(2)

Taking the first two fractions of (2) $\dfrac{dx}{1} = \dfrac{dy}{3}$ or $dy - 3dx = 0$

Integrating it , we get $y - 3x = c_1$ ......(3)

c_1 being an arbitrary constant.

Taking first and last fraction of (2) , $\dfrac{dx}{1} = \dfrac{dz}{5z + \tan(y - 3x)}$

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$ [Using (3)]

Integrating it , we get $x = \left(\dfrac{1}{5}\right) \log(5z + \tan c_1) + \left(\dfrac{1}{5}\right) c_2$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 5

where c_2 is an arbitrary constant.

or $\qquad 5x - \log\left[5z + \tan(y-3x)\right] = c_2 \qquad$ [using(3)] $\qquad$ ......(4)

From (3)and (4), the required general integral is

$$5x - \log\left[5z + \tan(y-3x)\right] = \phi(y-3x)$$

Exercise 2.3

Solve the following partial differential equations

1. $xzp + yzq = xy$

2. $zp - zq = x + y$

3. $z(p-q) = z^2 + (x+y)^2$

4. $p + 3q = z + \cot(y-3x)$

5. $py + qx = xy\, z^2 \left(x^2 - y^2\right)$

6. $p - 2q = 3x^2 \sin(y+2x)$

7. $xp - yq = xy$

Answers

In all the following answers ϕ being an arbitrary function :

1. $\phi\left(xy - z^2,\ \dfrac{x}{y}\right) = 0$

2. $2x(x+y) - z^2 = \phi(x+y)$

3. $e^{2y}\left[z^2 + (x+y)^2\right] = \phi(x+y)$

4. $x - \log\left|z + \cot(y-3x)\right| = \phi(y-3x)$

5. $y^2\left(x^2 - y^2\right) + \left(\dfrac{2}{z}\right) = \phi\left(x^2 - y^2\right)$

6. $x^3 \sin(y+2x) - z = \phi(y+2x)$

7. $x\, e^{-\frac{z}{xy}} = \phi(xy)$

2.4 Lagrange's Method based on type III

Type III : Lagrange auxiliary equations are $\qquad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} \qquad$ ......(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, each fraction in (1) will be equal to

$$\frac{P_1\, dx + Q_1\, dy + R_1\, dz}{P_1\, P + Q_1\, Q + R_1\, R} \qquad \text{......(2)}$$

If $P_1 P + Q_1 Q + R_1 R = 0$, then we know that the numerator of (2) is also zero. This gives

$P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. Another integral can be

obtained by repeating this method or the methods explained earlier.

Example : Solve $x(y^2 - z^2) p - y(z^2 + x^2)q = z(x^2 + y^2)$.

Solution : The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} \qquad ......(1)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$$

Integrating, we get $\quad x^2 + y^2 + z^2 = c_1 \qquad ......(2)$

Choosing $\dfrac{1}{x}$, $-\dfrac{1}{y}$, $-\dfrac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{\left(\dfrac{1}{x}\right)dx - \left(\dfrac{1}{y}\right)dy - \left(\dfrac{1}{z}\right)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{\left(\dfrac{1}{x}\right)dx - \left(\dfrac{1}{y}\right)dy - \left(\dfrac{1}{z}\right)dz}{0}$$

$$\Rightarrow \qquad \left(\frac{1}{x}\right)dx - \left(\frac{1}{y}\right)dy - \left(\frac{1}{z}\right)dz = 0$$

Integrating $\quad \log x - \log y - \log z = \log c_2$

$$\Rightarrow \qquad \log\left\{\frac{x}{yz}\right\} = \log c_2 \qquad \Rightarrow \qquad \frac{x}{yz} = c_2 \qquad ......(3)$$

∴ The required solution is $\phi\left(x^2 + y^2 + z^2, \ \dfrac{x}{yz}\right) = 0$.

Exercise 2.4

Solve the following partial differential equations :

1. $x(y^2 + z) p - y(x^2 + z)q = z(x^2 - y^2)$
2. $(z^2 - 2yz - y^2) p + (xy + zx)q = xy - zx$

3. $z(xp - yq) = y^2 - x^2$
4. $(x + 2z) p + (4zx - y)q = 2x^2 + y$

5. $\left(\dfrac{b-c}{a}\right)yzp + \left(\dfrac{c-a}{b}\right)zxq = \left(\dfrac{a-b}{c}\right)xy$
6. $(x - y) p + (x + y)q = 2xz$

7. $(3x + y - z) p + (x + y - z)q = 2(z - y)$
8. $x(y^2 - z^2) p + y(z^2 - x^2)q = z(x^2 - y^2)$

9. $(y - zx) p + (x + yz)q = x^2 + y^2$
10. $x^2(y - z) p + y^2(z - x)q = z^2(x - y)$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 7

Answers

1. $\phi\left(x^2 + y^2 - 2z,\ xyz\right) = 0$

2. $\phi\left(y^2 - z^2 - 2yz,\ x^2 + y^2 + z^2\right) = 0$

3. $\phi\left(x^2 + y^2 + z^2,\ xy\right) = 0$

4. $\phi\left(xy - z^2,\ x^2 - y - z\right) = 0$

5. $\phi\left(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2\right) = 0$

6. $\phi\left\{x + y - \log z,\ \left(x^2 + y^2\right)e^{-2\tan^{-1}\left(\frac{y}{x}\right)}\right\} = 0$

7. $\phi\left(x - 3y - z,\ \dfrac{x - y + z}{\sqrt{x + y - z}}\right) = 0$

8. $\phi\left(x^2 + y^2 + z^2,\ xyz\right) = 0$

9. $\phi\left(x^2 - y^2 + z^2,\ xy - z\right) = 0$

10. $\phi\left(xyz,\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right) = 0$

2.5 Lagrange's Method based on type IV

Type IV : Lagrange auxiliary equations are $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ ......(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, each fraction in (1) will be equal to

$$\frac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1 P + Q_1 Q + R_1 R} \qquad ......(2)$$

If the numerator of (2) is exact differential of the denominator of (2) then (2) can be combined with a suitable fraction in (1) to give an integral. Another integral can be obtained by repeating this method or the methods explained earlier.

Example : Solve $(x^2 - y^2 - yz)\,p + (x^2 - y^2 - zx)\,q = z(x - y)$.

Solution : Here Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x - y)} \qquad ......(1)$$

Choosing $1, -1, 0$ as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 - y^2 - yz) - (x^2 - y^2 - zx)} = \frac{dx - dy}{z(x - y)} \qquad ......(2)$$

Choosing $x, -y, 0$ as multipliers each fraction of (1)

$$= \frac{x\,dx - y\,dy}{x(x^2 - y^2 - yz) - y(x^2 - y^2 - zx)} = \frac{x\,dx - y\,dy}{(x - y)(x^2 - y^2)} \qquad ......(3)$$

From (1), (2), (3), we have

$$\frac{dz}{z(x-y)} = \frac{dx-dy}{z(x-y)} = \frac{x\,dx - y\,dy}{(x-y)(x^2 - y^2)}$$

or $$\frac{dz}{z} = \frac{dx-dy}{z} = \frac{2x\,dx - 2y\,dy}{2(x^2 - y^2)}$$ ......(4)

Taking the first two fractions of (4), we have

$$dz = dx - dy$$

Integrating, we get $$z - x + y = c_1$$ ......(5)

Again, taking the first and third fractions of (4) $$\frac{d(x^2 - y^2)}{(x^2 - y^2)} - \left(\frac{2}{z}\right)dz = 0$$

Integrating , we get $$\log (x^2 - y^2) - 2\log z = c_2 \quad \text{or} \quad \frac{(x^2 - y^2)}{z^2} = c_2$$ ......(6)

From (5) and (6), solution is $$\phi\left(z - x + y, \ \frac{(x^2 - y^2)}{z^2}\right).$$

Exercise 2.5

Solve the following partial differential equations :

1. $y^2(x-y)p + x^2(y-x)q = z(x^2 + y^2)$ 2. $(p+q)(x+y) = 1$

3. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ 4. $zp + zq = z^2 + (x-y)^2$

5. $xzp + yzq = xy$ 6. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$

7. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ 8. $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$

Answers

1. $\phi\left(x^3 + y^3, \ \dfrac{x-y}{z}\right) = 0$ 2. $\phi\left(y - x, \ e^{-2z}\,y + x\right) = 0$

3. $\phi\left(\dfrac{y}{z}, \ \dfrac{x^2 + y^2 + z^2}{z}\right) = 0$ 4. $\log\left[z^2 + (x-y)^2\right] - 2x = \phi(x-y)$

5. $\phi\left(\dfrac{x}{y}, \ xy - z^2\right) = 0$ 6. $\phi\left(yz + x^2, \ 2xz - y^2\right) = 0$

7. $\phi\left(\dfrac{x-y}{y-z}, \ \dfrac{z-x}{y-z}\right) = 0$ 8. $\phi\left(\dfrac{y-z}{x-y}, \ \dfrac{x-z}{x-y}\right) = 0$

2.6 Miscellaneous Examples on $Pp + Qq = R$

Example 1 : Solve $x\left(\dfrac{\partial u}{\partial x}\right) + y\left(\dfrac{\partial u}{\partial y}\right) + z\left(\dfrac{\partial u}{\partial z}\right) = xyz$.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 9

Solution : Here the auxiliary equation for the given equation are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz} \qquad \qquad ......(1)$$

Taking the first two fractions of (1)

$$\left(\frac{1}{x}\right)dx - \left(\frac{1}{y}\right)dy = 0$$

Integrating it, $\qquad \log x - \log y = \log C_1 \quad$ or $\qquad \dfrac{x}{y} = C_1 \qquad ......(2)$

Taking the first and third fractions of (1) , $\quad \left(\dfrac{1}{x}\right)dx - \left(\dfrac{1}{z}\right)dz = 0$

Integrating it, $\qquad \qquad \log x - \log z = \log C_2 \quad$ or $\quad \dfrac{x}{z} = C_2 \qquad ......(3)$

Choosing yz, zx, xy as multipliers, each fraction of (1)

$$= \frac{yz\,dx + zx\,dy + xy\,dz}{xyz + xyz + xyz} = \frac{d(xyz)}{3\,xyz} \qquad \qquad ......(4)$$

Combining the fourth fraction of (1) with fraction (4), we get

$$\frac{du}{xyz} = \frac{d(xyz)}{3\,xyz} \qquad \qquad \text{or} \qquad d(xyz) - 3du = 0$$

Integrating $\qquad \qquad xyz - 3u = C_3 \qquad \qquad ......(5)$

From (2), (3) and (5), the required general solution is $\phi\left(\dfrac{x}{y}, \dfrac{x}{z}, xyz - 3u\right) = 0$

Example 2 : Solve $(x_3 - x_2)p_1 + x_2 p_2 - x_3 p_3 + x_2^2 - (x_2 x_1 + x_2 x_3) = 0$.

Solution : Re-writing the given equation in the standard from, we get

$$(x_3 - x_2)p_1 + x_2 p_2 - x_3 p_3 = x_2 x_1 + x_2 x_3 - x_2^2 \qquad \qquad ......(1)$$

Here the auxiliary equations of (1) are $\dfrac{dx_1}{x_3 - x_2} = \dfrac{dx_2}{x_2} = \dfrac{dx_3}{-x_3} = \dfrac{dz}{x_2 x_1 + x_2 x_3 - x_2^2} \qquad ......(2)$

Taking the second and the third fractions of (2), we get

$$\left(\frac{1}{x_2}\right)dx_2 + \left(\frac{1}{x_3}\right)dx_3 = 0$$

so that $\qquad \qquad \log x_2 + \log x_3 = \log C_1 \quad$ or $\qquad x_2 x_3 = C_1 \qquad ......(3)$

Choosing 1, 1, 1, 0 as multipliers each fraction of (2)

$$= \frac{dx_1 + dx_2 + dx_3}{(x_3 - x_2) + x_2 - x_3} = \frac{dx_1 + dx_2 + dx_3}{0}$$

$\therefore \qquad\qquad dx_1 + dx_2 + dx_3 = 0$

Integrating $\qquad\qquad x_1 + x_2 + x_3 = C_2 \qquad\qquad\qquad\qquad ......(4)$

Choosing $x_2, x_1, 0, 0$ as multipliers each fraction of (2)

$$= \frac{x_2 \, dx_1 + x_1 dx_2}{x_2(x_3 - x_2) + x_1 x_2} = \frac{d(x_1 x_2)}{x_1 x_2 + x_2 x_3 - x_2^2} \qquad\qquad ......(5)$$

Combining the last fraction of (2) with fraction (5), we have

$$\frac{dz}{x_1 x_2 + x_2 x_3 - x_2^2} = \frac{d(x_1 \, x_2)}{x_1 x_2 + x_2 x_3 - x_2^2}$$

or $\qquad\qquad dz - d(x_1 \, x_2) = 0$ Integrating, $z - x_1 x_2 = C_3 \qquad\qquad ......(6)$

From (3), (4) and (6), the required general solution is $\phi(x_2 \, x_3, \, x_1 + x_2 + x_3, \, z - x_1 x_2) = 0$.

Exercise 2.6

1. $p_2 + p_3 = 1 + p_1$
   2. $x_2 \, x_3 \, p_1 + x_3 \, x_1 \, p_2 + x_1 \, x_2 \, p_3 + x_1 x_2 x_3 = 0$

Answers

1. $\phi(x_1 + x_2, \, x_1 + x_3, \, x_1 + z) = 0$
   2. $\phi(x_1^2 + 2z, \, x_1^2 - x_2^2, \, x_2^2 - x_3^2) = 0$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 1

Chapter - 3

3.1 Cauchy Problem

Integral surface passing through a given curve : Let $Pp + Qq = R$ 　　　　…..(1)

be the given equation. Let its auxiliary equations give the following two independent solutions

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ 　　　　…..(2)

We wish to obtain the integral surface which passes through the curve C whose equation in

parameteric form is given by $x = x(t),\ \ y = y(t),\ \ z = z(t)$ 　　　　…..(3)

Where t is a parameter. Then (2) be may be expressed as

$u\big(x(t),\ \ y(t),\ z(t)\big) = c_1$ and $v\big(x(t), y(t), z(t)\big) = c_2$ 　　　　…..(4)

We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2.

Finally we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

Example 1 : Find the integral surface of the linear PDE $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which

contains the straight line $x + y = 0,\ \ z = 1$

Solution : Given $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ 　　　　…..(1)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{x(y^2 + z)} = \dfrac{dy}{-y(x^2 + z)} = \dfrac{dz}{(x^2 - y^2)z}$ 　　　　….(2)

Now $\dfrac{dx}{x(y^2 + z)} = \dfrac{dy}{-y(x^2 + z)} = \dfrac{dz}{(x^2 - y^2)z} = \dfrac{xdx + ydy - dz}{0}$

$\Rightarrow\quad \dfrac{x^2}{2} + \dfrac{y^2}{2} - z = c_1\qquad \Rightarrow x^2 + y^2 - 2z = c_1$ 　　　　….(3)

Again, $\dfrac{dx}{x(y^2 + z)} = \dfrac{dy}{-y(x^2 + z)} = \dfrac{dz}{(x^2 - y^2)z} = \dfrac{\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz}{0}$

$\Rightarrow\quad \log x + \log y + \log z = \log c_2\qquad\quad \Rightarrow \log(xyz) = \log c_2$

$\Rightarrow\quad xyz = c_2$ 　　　　…..(4)

Now equation (1) passes through the curve $x + y = 0, z = 1$

Taking t as a parameter then equation of straight line in parameteric form is

$x = t, \; y = -t, \; z = 1$ …..(5)

Using (5) in (3) and (4), we have

$-t^2 = c_2 \;$ and $\; 2t^2 - 2 = c_1$ ….(6)

Eliminating t from (6), we get $-2c_2 - 2 = c_1 \;\Rightarrow\; c_1 + 2c_2 = -2$ …..(7)

Put values of c_1 and c_2 from (3) and (4) in (7), we get

$x^2 + y^2 - 2z + 2xyz = -2$

Example 2 : The integral surface of $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 + y^2$ passing through the curve

$x = 1 - t, \; y = 1 + t, \; z = 1 + t^2$ is

Solution : Given $yp + xq = x^2 + y^2$ …..(1)

Lagrange's A.E. is $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{x^2 + y^2}$

Now taking first two fractions we get $x^2 - y^2 = c_1$ ….(2)

Again $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{x^2 - y^2} = \dfrac{ydx + xdy - dz}{0} \;\Rightarrow\; ydx + xdy = dz$

$d(xy) = dz \quad \Rightarrow\; xy = z \neq c_2 \qquad \Rightarrow\; xy - z = c_2$ ….(3)

Now put $x = 1 - t, \; y = 1 + t, \; z = 1 + t^2$ in (2) and (3) we have,

$\cancel{1} + \cancel{t^2} - 2t - \cancel{1} - \cancel{t^2} - 2t = c_1 \qquad \Rightarrow\; t = -\dfrac{c_1}{4}$

Again (3) $\qquad \Rightarrow \qquad 1 - t^2 - 1 - t^2 = c_2 \qquad\qquad -2t^2 = c_2$

$\Rightarrow \qquad -2 \cdot \dfrac{c_1^2}{16} = c_2 \;\Rightarrow\; \dfrac{-c_1^2}{8} = c_2$

Put the values of c_1 and c_2, we get $xy - z = -\dfrac{1}{8}(x^2 - y^2)^2 \qquad \Rightarrow\; z = xy + \dfrac{1}{8}(x^2 - y^2)^2$

Exercise 3.1

1. Find the integral surface of the PDE $(x - y)p + (y - x - z)q = z$ through the circle $z = 1, \; x^2 + y^2 = 1$

2. Find the equation of integral surface of PDE $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ which passes through the line $x = 1, \; y = 0$

3. Find the general integral of PDE $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ and also the particular integral which passes through the line $x = 1, \; y = 0$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

4. Find the integral surface of $x^2 p + y^2 q + z^2 = 0$, which passes through the hyperbola

   $xy = x + y, \ z = 1$

5. Find the integral surface of PDE $yp + xq = z - 1$ which passes through the curve

   $z = x^2 + y^2, \ y = 2x$

Answers

1. $z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0$

2. $(x - y)(xy + yz + zx) + y - z = 0$ 3. $x^2 + y^2 + z - xz - y = 1$

4. $yz + 2xy + xz = 3xyz$ 5. $\dfrac{5}{3\sqrt{3}}(y^2 - x^2)^{\frac{1}{2}} = \dfrac{(z - 1)}{x + y}$

3.2 Existence and Uniqueness of integral surface passing through a given curve

Given the partial differential equation $Pp + Qq = R$ with initial curve

$$C : x_0(t), \ y_0(t), \ z_0(t) \qquad \qquad \text{.....(1)}$$

Then number of solutions of this PDE are according as

Unique Solution : If $\dfrac{P(x_0, y_0, z_0)}{\dfrac{dx_0}{dt}} \neq \dfrac{Q(x_0, y_0, z_0)}{\dfrac{dy_0}{dt}}$

$$\text{OR}$$

$$\Delta = \begin{vmatrix} P\big(x_0(t), \ y_0(t), z_0(t)\big) & Q\big(x_0(t), \ y_0(t), \ z_0(t)\big) \\ \dfrac{dx_0}{dt} & \dfrac{dy_0}{dt} \end{vmatrix} \neq 0$$

then (1) has unique solution.

No Solution : If $\dfrac{P(x_0, y_0, z_0)}{\dfrac{dx_0}{dt}} = \dfrac{Q(x_0, y_0, z_0)}{\dfrac{dy_0}{dt}} \neq \dfrac{R(x_0, y_0, z_0)}{\dfrac{dz_0}{dt}}$

then (1) has no solution.

Infinite Solution : If $\dfrac{P(x_0, y_0, z_0)}{\dfrac{dx_0}{dt}} = \dfrac{Q(x_0, y_0, z_0)}{\dfrac{dy_0}{dt}} = \dfrac{R(x_0, y_0, z_0)}{\dfrac{dz_0}{dt}} = u$

then (1) has infinite solutions.

Example 1 : Consider the Cauchy's problem $u_x - u_y = 2$ check whether equation has unique solution passing through the curve $(2s, s, 2s)$.

Solution : Given $P(x(s), y(s), z(s)) = 1$, $\qquad Q(x(s), y(s), z(s)) = -1$ and $R(x(s), y(s), z(s)) = 2$.

Given that $x_0(s) = 2s$, $y_0(s) = s$, $z_0(s) = 2s$

Now $\quad \Delta = \begin{vmatrix} P(x_0, y_0, z_0) & Q(x_0, y_0, z_0) \\ \dfrac{dx_0}{ds} & \dfrac{dy_0}{ds} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3 \neq 0$

then PDE has unique solution.

Remark : If $\Delta = 0$, then we cannot say anything about number of solutions.

Example 2 : Consider the Cauchy's problem $z_x = z$; $z(x, 0) = \sin x$. Find the number of solution passing through the curve $(s, 0, \sin s)$.

Solution : $z_x = z \qquad P = 1, \ Q = 0, \ R = z$

Now, $x_0(s) = s$, $y_0(s) = 0$, $z_0(s) = \sin s$

$\Delta = \begin{vmatrix} P(x_0, y_0, z_0) & Q(x_0, y_0, z_0) \\ \dfrac{dx_0}{ds} & \dfrac{dy_0}{ds} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$\Delta = 0$, we cannot say anything.

Now, $z_x = z \qquad \dfrac{dx}{1} = \dfrac{dy}{0} = \dfrac{dz}{z} \qquad \Rightarrow dy = 0 \qquad \Rightarrow y = c_1$

again, $\qquad \dfrac{dx}{1} = \dfrac{dz}{z} \qquad \Rightarrow x = \log z + \log c_2$

$\Rightarrow x = \log c_2 z \qquad \Rightarrow e^x = c_2 z \qquad \Rightarrow c_2 = z e^{-x}$

Then general solution is $ze^{-x} = \phi(y)$

Now solution passing through the curve $(s, 0, \sin s)$

$\therefore \qquad (\sin s) e^{-s} = \phi(0) \qquad \Rightarrow \sin s = \phi(0) e^s$

which is not possible.

$\Rightarrow \qquad$ No such ϕ exists and hence given PDE has no solution.

Exercise 3.2

1. If $u_x + u_y = u$ passes through $(s, s, \sin s)$, then find the number of solution of this PDE, $0 < s < 1$.

2. Find number of solution of $z_x = z$; $z(x, 0) = e^x$.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

3. If $z_x - z_y = 2$ passes through $(s, s, 2s)$, then find the number of solution of this PDE, $0 < s < 1$.

Answers

1. No solution   2. Infinite solution   3. Unique solution

3.3 Surface orthogonal to a given system of surfaces

Let $f(x, y, z) = c$ ……(1)

Represent a system of surfaces, where c is parameter. Suppose we wish to obtain a system of surfaces

which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to

(1) which passes through that point are $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$.

Let the surface $z = \phi(x, y)$ ……(2)

cut each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios

$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, -1$ i.e., $p, q, -1$

since normals at (x, y, z) to (1) and (2) are at right angles, we have $p\dfrac{\partial f}{\partial x} + q\dfrac{\partial f}{\partial y} - \dfrac{\partial f}{\partial z} = 0$

OR

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$ …..(3)

which is of the form $Pp + Qq = R$ conversely, we easily verify that any solution of (3) is orthogonal

surface of (1).

Example : Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$

orthogonally and which passes through the circle $x^2 + y^2 = 1, \; z = 1$

Solution : The given system of surfaces is given by $f(x, y, z) = \dfrac{z(x + y)}{3z + 1} = c$ ……(1)

$\therefore \qquad \dfrac{\partial f}{\partial x} = \dfrac{z}{3z + 1}, \quad \dfrac{\partial f}{\partial y} = \dfrac{z}{3z + 1}, \quad \dfrac{\partial f}{\partial z} = (x + y)\dfrac{1 \cdot (3z + 1) - z \cdot 3}{(3z + 1)^2} = \dfrac{x + y}{(3z + 1)^2}$

The required orthogonal surface is solution of $p\dfrac{\partial f}{\partial x} + q\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial z}$

or $\quad \dfrac{z}{3z+1}p + \dfrac{z}{3z+1}q = \dfrac{x+y}{(3z+1)^2}$

or $\quad z(3z+1)p + z(3z+1)q = x+y$ .....(2)

Lagrange's auxiliary equations for (2) are $\dfrac{dx}{z(3z+1)} = \dfrac{dy}{z(3z+1)} = \dfrac{dz}{x+y}$ .....(3)

Taking the first two fractions of (3), we get

$dx - dy = 0 \quad$ so that $x - y = c_1$ .....(4)

choosing $x, y, -z(3z+1)$ as multipliers each fraction of (3)

$= \dfrac{xdx + ydy - z(3z+1)dz}{0}$

$\therefore \quad xdx + ydy - 3z^2 dz - zdz = 0$

Integrating $\left(\dfrac{1}{2}\right)x^2 + \left(\dfrac{1}{2}\right)y^2 - 3\left(\dfrac{z^3}{3}\right) - \left(\dfrac{1}{2}\right)z^2 = \dfrac{1}{2}c_2$

or $\quad x^2 + y^2 - 2z^3 - z^2 = c_2$ ....(5)

Hence any surface which is orthogonal to (1) has equation of the form

$x^2 + y^2 - 2z^3 - z^2 = \phi(x-y)$ .....(6)

ϕ being an arbitrary function.

In order to get the desired surface passing through the circle $x^2 + y^2 = 1, \ z = 1$, we must choose

$\phi(x-y) = -2$. Thus, the required particular surface is $x^2 + y^2 - 2z^3 - z^2 = -2$

Exercise 3.3

1. Find the surface which is orthogonal to the one parameter system $z = cxy(x^2 + y^2)$ which passes through the hyperbola $x^2 - y^2 = a^2, \ z = 0$.

2. Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by $x(x^2 + y^2 + z^2) = c_1 y^2$.

3. Find the family of surfaces orthogonal to the family of surfaces given by the differential equation $(y+z)p + (z+x)q = x+y$.

Answers

1. $\left(x^2 + y^2 + 4z^2\right)^2 (x^2 - y^2)^2 = a^4(x^2 + y^2)$

2. $x^2 + y^2 + z^2 = z\phi\left(\dfrac{2x^2 + y^2}{z^2}\right)$

3. $xy + yz + zx = c$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Chapter - 4

4.1 Characteristic curve for Quasi linear partial differential equation

Consider a quasi linear PDE of the form $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$ ……(1)

Suppose that a solution z is known and consider the surface graph $z = z(x, y)$ in R^3.

A normal vector to this surface is given by $(p, q, -1)$.

As equation (1) is equivalent to the geometrical statement that the vector field

$(P(x, y, z), Q(x, y, z), R(x, y, z))$ is tangent to the surface $z = z(x, y)$ at every point, for the dot

product of this vector field with the above normal vector is zero. In other words, the graph of the

solution must be a union of integral curves of this vector field. These integral curves are called the

characteristic curves of original PDE.

The equation of the characteristic curve may be expressed invariantly by Lagrange charpit

equations.

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}$$

or, if a particular parametrization 't' of the curves is fixed, then these equations may be written as a

system of ordinary differential equation for $x(t)$, $y(t)$, $z(t)$;

$$\frac{dx}{dt} = P(x, y, z), \qquad \frac{dy}{dt} = Q(x, y, z), \qquad \frac{dz}{dt} = R(x, y, z)$$

These are characteristic equations for original system.

Example 1 : Find the characteristic curves for the PDE $z\, z_x + z_y = 1$ with

$$C : x_0 = s,\ y_0 = s,\ z_0 = \frac{s}{2},\ 0 \le s \le 1$$

Solution : Consider the PDE $z\, z_x + z_y = 1$ ……(1)

Then characteristic curve for (1) are $\dfrac{dx}{dt} = z$ ……(2)

$$\frac{dy}{dt} = 1$$ ..…(3)

$$\frac{dz}{dt} = 1$$ ..…(4)

with $C : x_0 = s, \ y_0 = s, \ z_0 = \dfrac{s}{2}$

Now $\dfrac{dz}{dt} = 1 \ \Rightarrow z = t + c_1$ ......(5)

From (2), $\dfrac{dx}{dt} = t + c_1 \qquad \Rightarrow \qquad x = \dfrac{t^2}{2} + c_1 t + c_2$ ......(6)

From (3), $\dfrac{dy}{dt} = 1 \qquad \Rightarrow \ y = t + c_3$ ......(7)

Now at $t = 0, \ x = s, \ y = s, \ z = s$ then from (5), (6), (7) we have

$\dfrac{s}{2} = 0 + c_1, \ \ s = \dfrac{0}{2} + c_1 \cdot 0 + c_2$ and $s = 0 + c_3$

$\Rightarrow c_1 = \dfrac{s}{2}, \ c_2 = s, \ c_3 = s$.

$\therefore$ characteristic curves are $x = \dfrac{t^2}{2} + \dfrac{s}{2} t + s \ , \qquad y = t + s \ , \qquad z = t + \dfrac{s}{2}$

Characteristic curve for semi linear : Consider the semi linear PDE is

$$P(x, y) p + Q(x, y) q = R(x, y, z)$$

Then characteristic curve are given by $\dfrac{dx}{P(x, y)} = \dfrac{dy}{Q(x, y)} \qquad \Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{Q(x, y)}{P(x, y)}$

Example : Find the characteristic curve for semi linear PDE $2y z_x + (2x + y^2) z_y = 0$ passing through $(0, 0)$.

Solution : Consider the semilinear PDE is $2y z_x + (2x + y^2) z_y = 0$ .....(1)

Then characteristic curve is $\dfrac{dy}{dx} = \dfrac{2x + y^2}{2y} \qquad \Rightarrow 2y \dfrac{dy}{dx} - y^2 = 2x$ .....(2)

Put $y^2 = t \qquad \Rightarrow \qquad 2y \dfrac{dy}{dx} = \dfrac{dt}{dx}$

Then (2) becomes $\dfrac{dt}{dx} - t = 2x$.

Then I.F. $= e^{\int -1 \cdot dx} = e^{-x}$

$\therefore$ solution of (3) is

$t\, e^{-x} = \int 2x e^{-x} \, dx \qquad = 2\left[(-x e^{-x}) + \int e^{-x} \, dx \right] + c_1 \quad = -2x e^{-x} - 2 e^{-x} + c_1$

$\Rightarrow t\, e^{-x} = -2 e^{-x}(x + 1) + c_1 \ \Rightarrow \ y^2 e^{-x} = -2 e^{-x}(x + 1) + c_1$ \qquad since it passes through $(0, 0)$

$\therefore \qquad 0 e^{-0} = -2 e^{-0}(0 + 1) + c_1 \qquad \Rightarrow 0 = -2 + c_1 \qquad \Rightarrow c_1 = 2$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

$\Rightarrow \quad y^2 e^{-x} = -2e^{-x}(x+1) + 2 \qquad \Rightarrow y^2 = -2x - 2 + 2e^x \qquad \Rightarrow y^2 = 2(e^x - x - 1)$

Which is the required characteristic curve.

Exercise 4.1

1. Find the characteristic curve for the PDE $zz_x + z_y = 0$ with $x_0 = s, \ y_0 = 0, \ z_0 = s, \ 0 \le s \le 1$.

2. Find the characteristic curve for PDE $zz_x + z_y = 0$ satisfies $z(x, 0) = \alpha + \beta x$.

3. Consider the partial differential equation $x\dfrac{\partial u}{\partial y} - y\dfrac{\partial u}{\partial x} = u$, then find the characteristic equation in (x, y) plane.

4. If $u(x, y)$ is a solution of PDE $x\dfrac{\partial u}{\partial y} - y\dfrac{\partial u}{\partial x} = u$ with $u(x, 0) = \sin\dfrac{\pi}{4}x$. Then find $u\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

Answers

1. $x = s(t+1), \ y = t, \ z = s$

2. $x = (\alpha + \beta s)t + s, \ y = t, \ z = \alpha + \beta s$

3. $x^2 + y^2 = c_1^2$ i.e., a circle centred at origin.

4. $u\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{\sqrt{2}} e^{\frac{\pi}{4}}$

J.R. INSTITUTE OF MATHEMATICS

---------------- **S C Q** ----------------

1. Equation $p \tan y + q \tan x = \sec^2 z$ is of order is

   1. 1   2. 2
   3. 0   4. None of these

2. Equation $\dfrac{\partial^2 z}{dx^2} - 2\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) + \left(\dfrac{\partial z}{\partial y}\right) = 0$ is of order

   1. 1   2. 2
   3. 3   4. None of these

3. The equation $(2x+3y)p + 4xq = x+y$ is

   1. linear   2. non-linear
   3. quasi-linear   4. semi-linear

4. $(x+y-z)\left(\dfrac{\partial z}{\partial x}\right) + (3x+2y)\dfrac{\partial z}{\partial y} + 2z = x+y$

   is

   1. linear   2. quasi-linear
   3. semi-linear   4. non-linear

5. The partial differential equation by eliminating arbitrary constant's a and b from the equation

   $(x+a)^2 + (y+b)^2 + z^2 = 1$

   is

   1. $z\left\{\left(\dfrac{\partial^2 z}{\partial x^2}\right) + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right\} = 1$

   2. $z^2\left\{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right\} = 1$

   3. $z\left\{\left(\dfrac{\partial^2 z}{\partial x^2}\right)^2 + \left(\dfrac{\partial^2 z}{\partial y^2}\right)^2 + 1\right\} = 1$

   4. $z^2\left\{\left(\dfrac{\partial^2 z}{\partial x^2}\right)^2 + \left(\dfrac{\partial^2 z}{\partial y^2}\right)^2 + 1\right\} = 1$

6. The partial differential equation by eliminating arbitrary constant a and b from the equation $z = ax + (1-a)y + b$ is

   1. $p + q^2 = 1$ 2. $p^2 + q = 1$
   3. $p^2 + q^2 = 1$ 4. $p + q = 1$

7. The partial differential equation by eliminating arbitrary function of f and g from the equation $v = \dfrac{\{f(r-at) + g(r+at)\}}{r}$

   1. $\dfrac{\partial^2 v}{\partial r^2} = \left(\dfrac{1}{a^2}\right) \times \left(\dfrac{\partial^2 v}{\partial t^2}\right) - \dfrac{2}{r}\dfrac{\partial v}{\partial r}$

   2. $\dfrac{\partial^2 v}{\partial r^2} = \dfrac{1}{a^2}\left(\dfrac{\partial^2 v}{\partial t^2}\right) + \dfrac{2}{r}\dfrac{\partial v}{\partial r}$

   3. $\dfrac{\partial^2 v}{\partial r^2} = \dfrac{1}{(a^2)}\left(\dfrac{\partial^2 v}{\partial t^2}\right) + \dfrac{2}{r}\dfrac{d^2 v}{dr^2}$

   4. $\dfrac{\partial^2 v}{\partial r^2} = \dfrac{1}{a^2}\left(\dfrac{\partial^2 v}{\partial t^2}\right) - \dfrac{2}{r}\dfrac{d^2 v}{dr^2}$

8. The partial differential equation by eliminating arbitrary function ϕ from $z + x - y = \phi(2 - x + y)$ is

   1. $p - q = 0$ 2. $p + q = 0$
   3. $p - q = 1$ 4. $p + q = 1$

9. The partial differential equation by eliminate a and b from $az + b = a^2 x + y$ is

   1. $p^2 q = 1$ 2. $pq = 1$
   3. $p^2 q^2 = 1$ 4. $pq^2 = 1$

10. The partial differential equation by eliminating the arbitrary function

    $\phi\left(x+y+z,\ x^2+y^2+z^2\right)=0$ is

    1. $(y-z)\,p-(z-x)\,q=x-y$

    2. $(y+z)\,p-(x+z)\,q=x+y$

    3. $(y-z)\,p+(z-x)\,q=x-y$

    4. $(y+z)\,p+(x+z)\,q=x+y$

11. Let $u(x,y)$ be the solution to the Cauchy problem

    $xu_x+u_y=1,\ u(x,0)=2\ln(x),\ x>1$ then

    $u(e,1)=$

    1. -1 2. 0

    3. 1 4. e

12. The solution of $xu_x+yu_y=0$ is of the form

    1. $f\left(\dfrac{y}{x}\right)$ 2. $f(x+y)$

    3. $f(x-y)$ 4. $f(xy)$

13. The general integral of PDE is

    $z\left(z^2+xy\right)\left(px-qy\right)=x^4$ is

    1. $xy=\phi\left(x^4+z^4+2xyz^2\right)$

    2. $\phi\left(xy,x^4-z^4-2xyz^2\right)=0$

    3. $\phi\left(xy,x^4+z^4+2xyz^2\right)=0$

    4. $xy=\phi\left(x^4+z^4-2xyz^2\right)$

14. Using the transformation $u=\dfrac{w}{y}$ in the partial differential equation $xu_x=u+yu_y$ the transformed equation has a solution of the form $w=$

    1. $f\left(\dfrac{x}{y}\right)$ 2. $f(x+y)$

    3. $f(x-y)$ 4. $f(xy)$

    (GATE 1997)

15. The partial differential equation of the family of surfaces $z=(x+y)+A(xy)$ is

    1. $xp-yq=0$ 2. $xp-yq=x-y$

    3. $xp+yq=x+y$ 4. $xp+yq=0$

    (GATE 1998)

16. The general integral of the partial differential equation

    $(y+zx)z_x-(x+yz)z_y=x^2-y^2$ is

    1. $F\left(x^2+y^2+z^2,\ xy+z\right)=0$

    2. $F\left(x^2+y^2-z^2,\ xy+z\right)=0$

    3. $F\left(x^2-y^2-z^2,\ xy+z\right)=0$

    4. $F\left(x^2+y^2+z^2,\ xy-z\right)=0$

    Where F is an arbitrary function.

    (GATE 2001)

17. The characteristic curves of the partial differential equation

    $(2x+u)u_x+(2y+u)u_y=u$, passing through

    $(1,1)$ for any arbitrary initial values prescribed on a non characteristic curve are given by

    1. $x=y$ 2. $x^2+y^2=2$

    3. $x+y=2$ 4. $x^2-xy+y^2=1$

    (GATE 2004)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

18. The integral surface of the partial differential equation $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ satisfying the condition $u(1, y) = y$ is given by

1. $u(x, y) = \dfrac{y}{x}$

2. $u(x, y) = \dfrac{2y}{x+1}$

3. $u(x, y) = \dfrac{y}{2-x}$

4. $u(x, y) = y + x - 1$ **(GATE 2005)**

19. If $u(x, y)$ is a solution of the equation

$x\dfrac{\partial u}{\partial y} - y\dfrac{\partial u}{\partial x} = u$ with $u(x,0) = \sin\left(\dfrac{\pi}{4}x\right)$,

then $u\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ equals

1. $\dfrac{1}{\sqrt{2}}e^{\frac{\pi}{4}}$ 2. $\dfrac{\pi}{4}e^{\frac{\pi}{\sqrt{2}}}$

3. $\dfrac{1}{\sqrt{2}}e^{\frac{1}{\sqrt{2}}}$ 4. $\dfrac{\pi}{4}e^{\frac{\pi}{4}}$ **(GATE 2006)**

20. Consider the partial differential equation

$\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ satisfying the initial

condition $u(x,0) = \alpha + \beta x$. If $u(x,t) = 1$

along the characteristic $x = t + 1$, then

1. $\alpha = 1, \beta = 1$ 2. $\alpha = 2, \beta = 0$

3. $\alpha = 0, \beta = 0$ 4. $\alpha = 0, \beta = 1$

(GATE 2006)

21. Let $u(x, y) = f(xe^{y}) + g(y^{2}\cos(y))$

where f and g are infinitely differentiable functions. Then the partial differential equation of minimum order satisfied by u is

1. $u_{xy} + xu_{xx} = u_{x}$

2. $u_{xy} + xu_{xx} = xu_{x}$

3. $u_{xy} - xu_{xx} = u_{x}$

4. $u_{xy} - xu_{xx} = xu_{x}$ **(GATE 2007)**

22. The initial value problem

$u_{x} + u_{y} = 1, u(s,s) = \sin s, 0 \le s \le 1$ has

1. Two solutions

2. A unique solution

3. No solution

4. Infinitely many solutions

(GATE 2008)

23. The characteristic curve of

$2yu_{x} + (2x + y^{2})u_{y} = 0$ passing through

$(0,0)$ is

1. $y^{2} = 2(e^{x} + x - 1)$

2. $y^{2} = 2(e^{x} - x + 1)$

3. $y^{2} = 2(e^{x} - x - 1)$

4. $y^{2} = 2(e^{x} + x + 1)$ **(GATE 2008)**

24. The solution of $xu_{x} + yu_{y} = 0$ is of the form

1. $f(y/x)$ 2. $f(x+y)$

3. $f(x-y)$ 4. $f(xy)$ **(GATE 2008)**

25. The integral surface satisfying the equation

    $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 + y^2$ and passing through

    the curve $x = 1-t,\ y = 1+t,\ z = 1+t^2$ is

    1. $z = xy + \dfrac{1}{2}\left(x^2 - y^2\right)^2$

    2. $z = xy + \dfrac{1}{4}\left(x^2 - y^2\right)^2$

    3. $z = xy + \dfrac{1}{8}\left(x^2 - y^2\right)^2$

    4. $z = xy + \dfrac{1}{16}\left(x^2 - y^2\right)^2$ **(GATE 2009)**

26. The integral surface for the Cauchy

    problem $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$ which passes through

    the circle $z = 0,\ x^2 + y^2 = 1$ is

    1. $x^2 + y^2 + 2z^2 + 2zx - 2yz - 1 = 0$

    2. $x^2 + y^2 + 2z^2 + 2zx + 2yz - 1 = 0$

    3. $x^2 + y^2 + 2z^2 - 2zx - 2yz - 1 = 0$

    4. $x^2 + y^2 + 2z^2 + 2zx + 2yz + 1 = 0$

    (GATE 2011)

27. The integral surface satisfying the partial

    differential equation $\dfrac{\partial z}{\partial x} + z^2\dfrac{\partial z}{\partial y} = 0$ and

    passing through the straight line

    $x = 1,\ y = z$ is

    1. $\left(x-1\right)z + z^2 = y^2$

    2. $x^2 + y^2 - z^2 = 1$

    3. $\left(y - z\right)x + x^2 = 1$

    4. $\left(x-1\right)z^2 + z = y$ **(GATE 2012)**

28. Let $a, b, c, d \in \mathbb{R}$ such that $c^2 + d^2 \neq 0$.

    Then, the Cauchy problem

$au_x + bu_y = e^{x+y},\ x, y \in \mathbb{R},\ u(x, y) = 0$ on

$cx + dy = 0$ has a unique solution if

1. $ac + bd \neq 0$

2. $ad - bc \neq 0$

3. $ac - bd \neq 0$

4. $ad + bc \neq 0$ **(GATE 2016)**

29. Let $u(x, y)$ be the solution of

    $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 4u$ satisfying the condition

    $u(x, y) = 1$ on the circle $x^2 + y^2 = 1$. Then

    $u(2, 2)$ equals _____.

    1. 25   2. 36

    3. 64   4. 81 **(GATE 2017)**

30. If $u(x, y) = 1 + x + y + f(xy)$, where

    $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function

    then u satisfies

    1. $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = x^2 - y^2$

    2. $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = 0$

    3. $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = x - y$

    4. $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = x - y$ **(GATE 2017)**

31. The Cauchy problem

    $\left.\begin{array}{ll} u_x(x, y) + u_y(x, y) = 0 & \text{for } (x, y) \in \mathbb{R}^2 \\ u(x, x) = 0 & \text{for all } x \in \mathbb{R} \end{array}\right\}$

    has

    1. A unique solution

    2. A family of straight lines as characteristics

    3. Solution which vanishes at $(2, 1)$

    4. Infinitely many solutions

    (CSIR NET SCQ June 2011)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

32. The solution of the Cauchy problem for the first order PDE $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$, on

$D = \left\{(x, y, z) \mid x^2 + y^2 \neq 0,\ z > 0\right\}$ with the

initial condition $x^2 + y^2 = 1,\ z = 1$ is

1. $z = x^2 + y^2$

2. $z = \left(x^2 + y^2\right)^2$

3. $z = \left(2 - \left(x^2 + y^2\right)\right)^{\frac{1}{2}}$

4. $z = \left(x^2 + y^2\right)^{\frac{1}{2}}$

(CSIR NET SCQ June 2013)

33. Consider the initial value problem

$\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0,\ u(0, y) = 4e^{-2y}$. Then the

value of $u(1,1)$ is

1. $4e^{-2}$ 2. $4e^{2}$

3. $2e^{-4}$ 4. $4e^{4}$

(CSIR NET SCQ June 2015)

34. Let $a, b \in \mathbb{R}$ such that $a^2 + b^2 \neq 0$. Then

the Cauchy problem $a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y} = 1$;

$x, y \in \mathbb{R},\ u(x, y) = x$ on $ax + by = 1$

1. has more than one solution if either a or b is zero.

2. has no solution

3. has a unique solution

4. has infinitely many solutions

(CSIR NET SCQ June 2015)

35. The solution of the initial value problem

$(x - y)\dfrac{\partial u}{\partial x} + (y - x - u)\dfrac{\partial u}{\partial y} = u,\ u(x,0) = 1$,

satisfies

1. $u^2(x - y + u) + (y - x - u) = 0$

2. $u^2(x + y + u) + (y - x - u) = 0$

3. $u^2(x - y + u) - (y + x + u) = 0$

4. $u^2(x - y + u) + (y + x - u) = 0$

(CSIR NET SCQ Dec 2015)

36. For the Cauchy problem

$u_t - uu_x = 0,\ x \in \mathbb{R},\ t > 0,\ u(x,0) = x,\ x \in \mathbb{R}$,

which of the following statements is true ?

1. The solution u exists for all $t > 0$

2. The solution u exists for $t < \dfrac{1}{2}$ and

    breaks down at $t = \dfrac{1}{2}$

3. The solution u exists for $t < 1$ and breaks down at $t = 1$

4. The solution u exists for $t < 2$ and breaks down at $t = 2$

(CSIR NET SCQ June 2016)

37. The solution of the partial differential equation $u_t - xu_x + 1 - u = 0,\ x \in \mathbb{R}, t > 0$

subject to $u(x,0) = g(x)$ is

1. $u(x,t) = 1 - e^{-t}\left(1 - g\left(xe^{t}\right)\right)$

2. $u(x,t) = 1 + e^{-t}\left(1 - g\left(xe^{t}\right)\right)$

3. $u(x,t) = 1 - e^{-t}\left(1 - g\left(xe^{-t}\right)\right)$

4. $u(x,t) = e^{-t}\left(1 - g\left(xe^t\right)\right)$

(CSIR NET SCQ June 2017)

38. The Cauchy problem

$$\left.\begin{array}{c} 2u_x + 3u_y = 5 \\ u = 1 \text{ on the line } 3x - 2y = 0 \end{array}\right\} \text{ has}$$

   1. exactly one solution
   2. exactly two solutions
   3. infinitely many solutions
   4. no solution

(CSIR NET June 2018)

-------------------- **M C Q** --------------------

-

1. A general solution of the PDE

   $uu_x + yu_y = x$ is of the form

   1. $f\left(u^2 - x^2, \dfrac{y}{x+u}\right) = 0$, where

      $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and $\nabla f \neq (0,0)$ at

      every point.

   2. $u^2 = g\left(\dfrac{y}{x+u}\right) + x^2$, $g \in C^1(\mathbb{R})$

   3. $f\left(u^2 + x^2\right) = 0$, $f \in C^1(\mathbb{R})$

   4. $f\left(x + y\right) = 0$, $f \in C^1(\mathbb{R})$

(CSIR NET MCQ June 2011)

2. The differential equation $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

   satisfying the initial condition

   $y = xg(x)$, $u = f(x)$ with

   1. $f(x) = 2x$, $g(x) = 1$ has no solution
   2. $f(x) = 2x^2$, $g(x) = 1$ has infinite

      number of solutions

   3. $f(x) = x^3$, $g(x) = x$ has a unique

solution

   4. $f(x) = x^4$, $g(x) = x$ has a unique

      Solution

(CSIR NET MCQ Dec 2011)

3. The Cauchy problem $xu_x + yu_y = 0$

   $u(x,y) = x$ on $x^2 + y^2 = 1$ has

   1. A solution for all $x \in \mathbb{R}$, $y \in \mathbb{R}$
   2. An unique solution in

      $\left\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\right\}$

   3. A bounded solution in

      $\left\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\right\}$

   4. An unique solution is

      $\left\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\right\}$ by the

      solution is unbounded

(CSIR NET MCQ June 2012)

4. Let $xyu = c_1$ and $x^2 + y^2 - 2u = c_2$, where

   c_1 and c_2 are arbitrary constants, be the

   first integrals of the PDE

   $x\left(u + y^2\right)\dfrac{\partial u}{\partial x} - y\left(u + x^2\right)\dfrac{\partial u}{\partial y} = \left(x^2 - y^2\right)u$.

   Then the solution of the PDE with

   $x + y = 0$, $u = 1$ is given by

   1. $x^3 + y^3 + 2xyu^2 + 2x^2u = 0$
   2. $x^3 + yx^2 + \left(x^2 + xy\right)u = 0$
   3. $x^2 + y^2 + 2\left(xy - 1\right)u + 2 = 0$
   4. $x^2 - y^2 - u\left(x + y - 2\right) - 2 = 0$

(CSIR NET MCQ June 2014)

5. The initial value problem

   $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = x$, $0 \leq x \leq 1$, $t > 0$ and

   $u(x,0) = 2x$ has

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com **Mob. 8607383607, 9802177766**

Page 7

1. A unique solution $u(x,t)$ which $\rightarrow \infty$ as $t \rightarrow \infty$

2. More than one solution

3. A solution which remains bounded as $t \rightarrow \infty$

4. No solution
   (CSIR NET MCQ June 2014)

6. Consider the Cauchy problem of finding $u = u(x,t)$ such that $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ for

   $x \in \mathbb{R}, t > 0, \; u(x,0) = u_0(x), x \in \mathbb{R}$.

   Which choice(s) of the following functions for u_0 yield a C^1 solution $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$

   1. $u_0(x) = \dfrac{1}{1+x^2}$

   2. $u_0(x) = x$

   3. $u_0(x) = 1+x^2$

   4. $u_0(x) = 1+2x$

   (CSIR NET MCQ Dec 2014)

7. The Cauchy problem $\left.\begin{array}{c} y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0 \\ u = g \text{ on } \Gamma \end{array}\right\}$.

   Has unique solution is neighbourhood of Γ for every differential function $g:\Gamma \rightarrow \mathbb{R}$ if

   1. $\Gamma = \{(x,0): x > 0\}$

   2. $\Gamma = \{(x,y): x^2 + y^2 = 1\}$

   3. $\Gamma = \{(x,y): x + y = 1, x > 1\}$

   4. $\Gamma = \{(x,y): y = x^2, x > 0\}$

   (CSIR NET MCQ Dec 2016)

8. Consider the partial differential equation $x\dfrac{\partial u}{\partial x} + yu\dfrac{\partial u}{\partial y} = -xy$ for $x > 0$ subject to $u = 5$ on $xy = 1$. Then

   1. $u(x,y)$ exists when $xy \leq 19$ and $u(x,y) = u(y,x)$ for $x > 0$; $y > 0$

   2. $u(x,y)$ exists when $xy \geq 19$ and $u(x,y) = u(y,x)$ for $x > 0$; $y > 0$

   3. $u(1,11) = 3, u(13,-1) = 7$

   4. $u(1,-1) = 5, u(11,1) = -5$
      (CSIR NET MCQ June 2017)

9. Let a be a fixed real constant. Consider the first order partial differential equation $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, x \in \mathbb{R}, t > 0$ with the initial data $u(x,0) = u_0(x), x \in \mathbb{R}$ where u_0 is a continuously differentiable function.

   Consider the following two statements.

   S_1 : There exists a bounded function u_0 for which the solution u is unbounded

   S_2 : If u_0 vanishes outside a compact set then for each fixed $T > 0$ there exists a compact set $K_T \subset \mathbb{R}$ such that $u(x,T)$ vanishes for $x \notin K_T$.

   Which of the following are true ?

   1. S_1 is true and S_2 is false

   2. S_1 is true and S_2 is also true

3. S_1 is false and S_2 is true

4. S_1 is false and S_2 is also false

(CSIR NET June 2018)

Answer Key

SCQ

1. 1	2. 2	3. 1
4. 2	5. 2	6. 4
7. 1	8. 2	9. 2
10. 3	11. 3	12. 1
13. 2	14. 4	15. 2
16. 2	17. 1	18. 1
19. 1	20. 4	21. 3
22. 4	23. 3	24. 1
25. 3	26. 3	27. 4
28. 1	29. 3	30. 3
31. 2	32. 4	33. 2
34. 3	35. 2	36. 3
37. None	38. 4	

MCQ

1. 1,2	2. 1,2,3,4	3. 2,3
4. 3	5. 3	6. 2,4
7. 1,3,4	8. 1,3	9. 3

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

Chapter - 5

5.1 Non-Linear Partial Differential equations of the first order

Solution of a Partial differential equation : A solution or integral of a partial differential equation is a relation between the variables, by means of which the partial derivatives obtained there from the partial differential equation is satisfied.

Now we discuss various classes of integrals or solutions of a partial differential equation of order one.

Def. Complete integral (C.I.) or Complete solution (C.S.) : Let z be a function of two independent variables x and y defined by

$$\phi(x, y, z, a, b) = 0 \qquad \qquad ......(1)$$

where a and b are arbitrary constants

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial \phi}{\partial x} + p\, \frac{\partial \phi}{\partial z} = 0 \qquad \qquad ......(2)$$

and

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0 \qquad \qquad ......(3)$$

Since there are two arbitrary a and b connected by the above three equations so eliminating a and b from these three equations, we get a relation of the form

$$f(x, y, z, p, q) = 0 \qquad \qquad ......(4)$$

which is a partial differential equation of order one.

Now suppose that the relation (1) has been derived from (4), by using some method, then the integral (1), which contains as many arbitrary constants as there are independent variables, is called the complete integral or complete solution of (4).

Def. Particular Integral (P.I.) or Particular Solution (P.S.) : A solution obtained by giving some particular values to the arbitrary constants in the complete solution of a partial differential equation of first order is called a particular solution of the given equation.

Def. Singular Integral (S.I.) or Singular Solution (S.S.) : Let $\phi(x, y, z, a, b) = 0$ be the complete solution of a partial differential equation $f(x, y, z, p, q) = 0$. Then the relation between x , y and z obtained by eliminating the arbitrary constants a and b between the equation.

$\phi(x, y, z, a, b) = 0$, $\dfrac{\partial \phi}{\partial a} = 0$, $\dfrac{\partial \phi}{\partial b} = 0$ is called the singular solution (or singular integral) of the equation $f(x, y, z, p, q) = 0$ provided it satisfies this equation.

Remarks : (i) The singular solution represents the envelope of the surfaces represented by the complete solution of the given partial differential equation.

(ii) In general the singular solution is distinct from the complete integral. however, in exceptional cases it may be contained in the complete integral, that is singular solution may be obtained by giving particular values to the constants in the complete solution. Since other solutions may appear in the process of obtaining the singular solution, it is necessary to test whether the singular solution satisfies the given partial differential equation.

Def. General Integral (G.I.) or General Solution (G.S) : Let $\phi(x, y, z, a, b) = 0$ be the complete solution of a partial differential equation $f(x, y, z, p, q) = 0$. Assume that in the complete solution, one of the constants is a function of the other say $b = \psi(a)$. Then the complete solution becomes.

$$\phi(x, y, z, a, \psi(a)) = 0 \qquad\qquad\qquad .....(1)$$

which represents one – parameter family of the surfaces of $f(x, y, z, p, q) = 0$.

The solution between x , y and z obtained by eliminating the arbitrary constant a between the equations (1) and $\dfrac{\partial \phi}{\partial a} = 0$ is called the general solution of the equation $f(x, y, z, p, q) = 0$.

Remark : If $b = \psi(a)$, where ψ is an arbitrary function, then the elimination of a between the equations $\phi(x, y, z, a, \psi(a)) = 0$ and $\dfrac{\partial \psi}{\partial a} = 0$ is not possible. Thus the general solution of the equation $f(x, y, z, p, q) = 0$ is written as the set of equations $\phi(x, y, z, a, \psi(a)) = 0$ and $\dfrac{\partial \psi}{\partial a} = 0$, where ϕ is any arbitrary function.

Important Note : While solving a non-linear partial differential equation, we must also find the singular and general solutions along with the complete solution. In the absence of singular and general solutions, only the complete solution is considered to be incomplete solution of the given partial differential equation.

However if you are asked to find complete solution of a given equation, then there is no need to give singular and general solution.

We shall use the following standard notations :

$$\frac{\partial f}{\partial x} = f_x, \ \frac{\partial f}{\partial y} = f_y, \ \ \frac{\partial f}{\partial z} = f_z, \ \ \frac{\partial f}{\partial p} = f_p, \ \ \frac{\partial f}{\partial q} = f_q$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

Therefore, Charpit's auxiliary equations may be written as

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

When the given partial differential equation is of some special form, then the Charpit's equations are simplified and the required solution is obtained very easily. In this regard we have the following types.

Type (I) : Equations containing only p and q :

Let the partial differential equation of first order and containing only p and q be

$$f(p, q) = 0 \qquad \qquad ......(1)$$

Charpit's auxiliary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_a} = \frac{dp}{f_x + p \cdot f_z} = \frac{dq}{f_y + qf_z}$$

Since f contains only p and q, we have $f_x = f_y = f_z = 0$, and so Charpit's equations takes the form

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$$\Rightarrow \qquad dp = 0 \qquad \text{and} \qquad dq = 0$$

$$\Rightarrow \qquad p = a \qquad \text{and} \qquad q = b \text{ , where } a \text{ and } b \text{ are constants}$$

Now (1) $\qquad \Rightarrow \qquad f(a, b) = 0 \qquad \Rightarrow \qquad b = \phi(a)$

Further, we have $\qquad dz = p\,dx + q\,dy$

$$\Rightarrow \qquad z = px + qy + c, \qquad \text{where } c \text{ is an arbitrary constant}$$

$$\Rightarrow \qquad z = ax + by + c$$

$$\Rightarrow \qquad z = ax + \phi(a)y + c \qquad \qquad .....(2)$$

where $(a, \phi(a)) = 0$ and a, c are arbitrary constants

To find the singular solution :

Let $\qquad \qquad F(x, y, z, a, c) = z - ax - \phi(a)y - c = 0 \qquad \qquad ......(3)$

be the above obtained complete integral of (1).

The singular solution of (1) is obtained be eliminating constants a and c from the equation

$$F = 0, \quad \frac{\partial F}{\partial a} = 0 \quad \text{and} \quad \frac{\partial F}{\partial c} = 0$$

i.e., $\qquad z - ax - \phi(a) y - c = 0, \quad -x - \phi'(a) y = 0 \quad$ and $\quad -1 = 0$

This is impossible because $-1 \neq 0$.

Hence there is no singular solution.

To find the general solution : Let $c = \psi(a)$ where ψ is an arbitrary function, so that (3) becomes

$$F(x, y, z, a) = z - ax - \phi(a) y - \psi(a)$$

Now, the general solution of (1) is obtained by eliminating the constant 'a' from the equations

$F = 0$ and $\dfrac{\partial F}{\partial a} = 0$.

i.e., $\qquad z - ax - \phi(a) y - \psi(a) = 0 \quad$ and $\quad -x - \phi'(a) y - \psi'(a) = 0$

Working Rule :

(i)   Take complete solution as $z = ax + by + c$ where a and c are arbitrary constants.

(ii)   Find $p = \dfrac{\partial z}{\partial x} = a, \ q = \dfrac{\partial z}{\partial y} = b$

(iii) Substitute the values of p and q in $f(p, q) = 0$ and find the value of b in terms of a. Put the value of $\phi(a)$ in the complete solution $z = ax + \phi(a) y + c$.

(iv) For general solution take $f(x, y, z, a, c) = z - ax - \phi(a) y - c$ and $c = \psi(a)$. Differentiate F partially w.r.t. a and write the general solution as $f(x, y, z, a, \psi(a)) = 0, \dfrac{\partial f}{\partial a} = 0$, i.e.,

$z - ax - \phi(a) y - \psi(a) = 0, -x - \phi'(a) y - \psi'(a) = 0$, where ψ is any arbitrary function.

(v)   Equation of the form $f(p, q) = 0$ has no singular solution.

Example : Solve $pq = k$, where k is a constant.

Solution : The given differential equation is $\quad pq = k \qquad$ ......(1)

which is of the form $f(p, q) = 0$, its solution is given by

$$z = ax + by + c \qquad \qquad ......(2)$$

where $ab = k$ or $b = \dfrac{k}{a}$, obtained by putting a for p and b for q in (1)

$\therefore$ From (2), the complete integral is

$$z = ax + \left(\dfrac{k}{a}\right) y + c \qquad \qquad ......(3)$$

where a and c are arbitrary constants.

For singular solution, differentiating (3) partially with respect to a and c, we get

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

$$0 = x - \left(\frac{k}{a^2}\right)y \text{ and } 0 = 1$$

But $0 = 1$ is absurd. Hence singular solution of (1) does not exist.

For general solution, putting $c = \phi(a)$ in (3), we get

$$z = ax + \left(\frac{k}{a}\right)y + \phi(a) \qquad \qquad ......(4)$$

Differentiating (4) partially with respect to 'a', we get

$$0 = x - \left(\frac{k}{a^2}\right)y + \phi'(a) \qquad \qquad ......(5)$$

Eliminating a from (4) and (5), we get the required general solution.

Exercise 5.1

Obtain the complete integral of the following equations :

1. $q = 4p^3$
2. $p^2 + q^2 = 4$

Solve the following partial differential equations

3. $p = q^2$
4. $p^2 - q^2 = 1$
5. $p = e^q$

6. $p + q = pq$
7. $p^2 + p = q^2$
8. $p^2 q^3 = 1$

9. $p^2 + q^2 = npq$; $|n| \geq 2$
10. $p^2 + 6p + 2q + 4 = 0$

Answers

1. $z = ax + 4a^3 y + c$
2. $z = ax + \sqrt{4 - a^2}\, y + c$

3.   C.S. $z = ax + \sqrt{a}\, y + c$, where a and c are arbitrary constants and $a \geq 0$

   S.S.   No singular solution.

   G.S. $z - ax - \sqrt{a}\, y - \psi(a) = 0, -x - \dfrac{1}{2\sqrt{a}} y - \psi'(a) = 0$ where ψ is any arbitrary function.

4.   C.S. $z = ax + \sqrt{a^2 - 1}\, y + c$, where a and c are arbitrary constants and $|a| \geq 1$.

   S.S.   No singular solution.

   G.S. $z - ax - \sqrt{a^2 - 1}\, y - \psi(a) = 0, -x - \dfrac{a}{\sqrt{a^2 - 1}} y - \psi'(a) = 0$ where ψ is any arbitrary function.

5.   C.S. $z = ax + y \log a + c$, where a and c are arbitrary constants are $a > 0$

     S.S.   No singular solution.

     G.S. $z - ax - y \log a - \psi(a) = 0$, $-x - \dfrac{y}{a} - \psi'(a) = 0$, where ψ is any arbitrary function.

6.   C.S. $z = ax + \dfrac{a}{a-1} y + c$, where a and c are arbitrary constants and $a \neq 1$

     S.S.   No singular solution.

     G.S. $z - ax - \dfrac{a}{a-1} y - \psi(a) = 0$, $-x + \dfrac{1}{(a-1)^2} y - \psi'(a) = 0$ where ψ is any arbitrary function.

7.   C.S. $z = ax + \sqrt{a^2 + a}\ y + c$, where a and c are arbitrary constants and $a \in R - (-1, 0)$.

     S.S.   No singular solution.

     G.S. $z - ax - \sqrt{a^2 + a}\ y - \psi(a) = 0$, $-x - \dfrac{2a+1}{2\sqrt{a^2 + a}}\ y - \psi'(a) = 0$ where a and c are arbitrary function

8.   C.S. $z = ax + a^{-\frac{2}{3}y} + c$, where a and c are arbitrary constants and $a \neq 0$

     S.S.   No singular solution.

     G.S. $z - ax - a^{-\frac{2}{3}} y - \psi(a) = 0$, $-x + \dfrac{2}{3} a^{-\frac{5}{3}} y - \psi'(a) = 0$, where ψ is any arbitrary function.

9.   C.S. $z = ax + \dfrac{a}{2}\left(n + \sqrt{n^2 - 4}\right) y + c$, where a and c are arbitrary constants.

     S.S.   No singular solution.

     G.S. $z - ax - \dfrac{a}{2}\left(n + \sqrt{n^2 - 4}\right) y - \psi(a) = 0$, $-x - \dfrac{1}{2}\left(n + \sqrt{n^2 - 4}\right) y - \psi'(a) = 0$ where ψ is any arbitrary function.

10.   C.S. $z = ax - \left(\dfrac{a^2}{2} + 3a + 2\right) y + c$, where a and c are arbitrary constants.

     S.S.   No singular solution

     G.S. $z - ax + \left(\dfrac{a^2}{2} + 3a + 2\right) y - \psi(a) = 0$, $-x + (a + 3) y - \psi'(a) = 0$, where ψ is any arbitrary function.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com Mob. 8607383607, 9802177766

Page 7

5.2 Equations reducible to type (I)

Here we discuss a special method to solve the non linear partial differential equations containing several variables. By using some transformations these equations can be reduced to standard type-(I) and after that these equations are easy to solve.

Example : Find complete integral of $z^2 p^2 y + 6 zpxy + 2 z qx^2 + 4x^2 y = 0$.

Solution : Rewrite the given differential equation as

$$z^2 y \left(\frac{\partial z}{\partial x} \right)^2 + 6zxy \left(\frac{\partial z}{\partial x} \right) + 2zx^2 \left(\frac{\partial z}{\partial y} \right) + 4x^2 y = 0$$

Dividing throughout by $x^2 y$, $\left(\frac{z}{x} \frac{\partial z}{\partial x} \right)^2 + 6 \left(\frac{z}{x} \frac{\partial z}{\partial x} \right) + 2 \left(\frac{z}{y} \frac{\partial z}{\partial y} \right) + 4 = 0$ .....(1)

Put $\qquad x\, dx = dX$, $\qquad y\, dy = dY$ $\qquad$ and $\qquad z\, dz = dZ$ ......(2)

so that $\qquad \dfrac{x^2}{2} = X$, $\qquad \dfrac{y^2}{2} = Y$ $\qquad$ and $\qquad \dfrac{z^2}{2} = Z$ ......(3)

Using (2) in (1), we get $\qquad \left(\dfrac{\partial Z}{\partial X} \right)^2 + 6 \left(\dfrac{\partial Z}{\partial X} \right) + 2 \left(\dfrac{\partial Z}{\partial Y} \right) + 4 = 0$

$\qquad$ or $\qquad P^2 + 6P + 2Q + 4 = 0$, where $\qquad P = \dfrac{\partial Z}{\partial X}$, $\qquad Q = \dfrac{\partial Z}{\partial Y}$ ......(4)

Equation (4) is of the form $f(P, Q) = 0$ Note that now we have P, Q, X, Y, Z in place of p, q, x, y, z in usual equations. Accordingly, solution of (4) is

$$Z = aX + bY + c \qquad\qquad ......(5)$$

where $\qquad a^2 + 6a + 2b + 4 = 0$ $\quad$ or $\quad b = -\left(a^2 + 6a + 4 \right) / 2$,

obtained by putting a for P and b for Q in (4).

So from (5), the required complete integral is

$$Z = aX - \left\{ \frac{(a^2 + 6a + 4)}{2} \right\} Y + c \text{, where } a \text{ and } c \text{ arbitrary constants.}$$

$\qquad$ or $\qquad \dfrac{z^2}{2} = a \left(\dfrac{x^2}{2} \right) - \left(a^2 + 6a + 4 \right) \times \left(\dfrac{y^2}{4} \right) + c$ $\qquad\qquad$ [Using (3)]

or $\qquad z^2 = ax^2 - \left(2 + 3a + \dfrac{a^2}{2}\right) y^2 + c'$, where $c' = 2c$

which is required complete integral.

Exercise 5.2

Find the complete integral of the following equation :

1. $p^2 x + q^2 y = z$ $\qquad\qquad$ 2. $p^2 x^2 + q^2 y^2 = z^2$

3. $p^2 - q^2 = z$ $\qquad\qquad\qquad$ 4. $pq = x^m y^n z^l$

5. $zy^2 p = xy^2 + xz^2 q^2$ $\qquad\qquad$ 6. $z^2 \left(\dfrac{p^2}{x^2} + \dfrac{q^2}{y^2}\right) = 1$

Answers

1. $2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\ \sqrt{1 - a^2} + c$ $\qquad$ 2. $\log z = a\log x + \sqrt{1 - a^2}\ \log y + c$

3. $2\sqrt{z} = ax \pm (\sqrt{a^2 - 1})y + c$ $\qquad$ 4. $\dfrac{2}{2 - l} z^{-\frac{l}{2} + 1} = \dfrac{x^{m+1}}{m+1} a + \dfrac{y^{n+1}}{(n+1)a} + c$

5. $z^2 = ax^2 \pm \sqrt{a - 1}\ y^2 + c$ $\qquad$ 6. $z^2 = ax^2 \pm \sqrt{1 - a^2}\ y^2 + c,\ -1 \le a \le 1$

5.3 Type (II) : Clairaut equation i.e., equations of the form $z = px + qy + f(p, q)$

Consider the Clairaut equation

$$z = px + qy + f(p, q) \qquad\qquad\qquad ......(1)$$

Let $\qquad\qquad F(x, y, z, p, q) = z - px - qy - f(p, q) \qquad\qquad ......(2)$

Charpit's auxiliary equations are

$$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{dz}{-pF_p - qF_a} = \frac{dp}{F_x + p \cdot F_z} = \frac{dq}{F_y + qF_z}$$

or $\qquad \dfrac{dx}{-x - f_p} = \dfrac{dy}{-y - f_q} = \dfrac{dz}{-px - qy - pf_p - qf_q} = \dfrac{dp}{-p + p} = \dfrac{dq}{-q + q}$

The last two fractions gives

$$dp = 0 \quad \text{and} \quad dq = 0$$

$\Rightarrow \qquad\qquad p = a \quad \text{and} \quad q = b \quad$ where a, b are arbitrary constants

Substituting these values in (1), the complete integral is given by

$$z = ax + by + f(a, b)$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 9

To find the singular solution :

Let $\qquad\qquad\qquad\qquad F\left(x,\,y,\,z,\,a,\,b\right)=z-ax-by-f\left(a,b\right)$ $\qquad$......(3)

The singular solution of (1) is obtained by eliminating constant a and b from the equations

$$F(x,y,z,a,b)=0 \;,\; \frac{\partial F}{\partial a}=0 \quad \text{and} \quad \frac{\partial F}{\partial b}=0\,,$$

i.e.,

$$z-ax-by-f(a,b)=0\;,\; -x-\frac{\partial f}{\partial a}=0 \quad \text{and} \quad -y-\frac{\partial f}{\partial b}=0$$

provided it satisfies the given equation

To find the general solution : Let $b=\psi(a)$, where ψ is any arbitrary function, so that (3) becomes

$$F\left(x,\,y,\,z,\,a,\,\psi\left(a\right)\right)=z-ax-\psi\left(a\right)y-f\left(a,\,\psi\left(a\right)\right)$$

Now the general solution of (1) is obtained by eliminating the constant 'a' from the equations

$$F=0 \;\; \text{and} \;\; \frac{\partial F}{\partial a}=0$$

i.e., $\qquad\qquad z-ax-\psi(a)\,y-g(a,\psi(a))=0 \quad \text{and} \quad -x-\psi'(a)\,y-f(a,\psi(a))=0$

Working Rules for solving $z = px + qy + g(p, q)$:

(i) Take complete solution as $z=ax+by+g(a,b)$ where a and b are arbitrary constants.

(ii) For singular solution, take $f(x,y,z,a,b)=z-ax-by-g(a,b)$. Find $\dfrac{\partial f}{\partial a}$ and $\dfrac{\partial f}{\partial b}$. Eliminate a

and b from the equations $f=0, \dfrac{\partial f}{\partial a}=0, \dfrac{\partial f}{\partial b}=0$. This gives the singular solution.

(iii) For general solution, take $b=\psi(a)$, where ψ is any arbitrary function. The equations

$f=0, \dfrac{\partial f}{\partial a}=0$ constitute the general solution.

Example : Solve $z=px+qy+pq$.

Solution : The complete integral of the given equation is $z=ax+by+ab$ $\qquad$......(1)

where a,b being arbitrary constants.

Singular integral : Differentiating (1) partially w.r.t. a and b, we have

$$0=x+b \qquad \text{and} \qquad 0=y+a \qquad\qquad ......(2)$$

Eliminating a and b between (1) and (2), we get

$$z=-xy-xy+xy \qquad \text{i.e.,} \;\; z=-xy$$

which is the required singular solution, for it satisfies the given equation.

General Integral : Take $b = \phi(a)$, here ϕ denotes an arbitrary function.

Then (1) becomes $\qquad\qquad z = ax + \phi(a)y + a\phi(a)$ $\qquad\qquad$ ......(3)

Differentiating (3) partially w.r.t. a $\quad 0 = x + \phi'(a)y + \phi(a) + a\phi'(a)$ $\qquad$ ......(4)

The general integral is obtained by eliminating a between (3) and (4).

Exercise 5.3

Find the complete integral of the following equations :

1. $(p+q)(z - px - qy) = 1$ $\qquad\qquad$ 2. $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

3. $2q(z - px - qy) = 1 + q^2$

Solve the following partial differential equations :

4. $z = px + qy + 5pq$ $\qquad\qquad$ 5. $z = px + qy + \dfrac{p}{q}$

6. $z = px + qy + \log(pq)$ $\qquad\qquad$ 7. $z = px + qy + p^2 - q^2$

8. $z = px + qy + p^2 q^2$ $\qquad\qquad$ 9. $z = px + qy + 3(pq)^{\frac{1}{3}}$

Answers

1. $z = ax + by + \dfrac{1}{a+b}$ $\qquad$ 2. $z = ax + by + \dfrac{(a^4 + b^4)}{ab}$ $\qquad$ 3. $z = ax + by + \dfrac{1 + b^2}{2b}$

4. C.S. $z = ax + by + 5ab$, $\qquad\qquad$ S.S. $\quad 5z + xy = 0$

   G.S. $z - ax - \psi(a)y - 5a\psi(a) = 0$, $x + 5\psi(a) + (y + 5a)\psi'(a) = 0$

5. C.S. $z = ax + by + \dfrac{a}{b}$, $\qquad\qquad$ S.S. $\quad xz + y = 0$.

   G.S. $z - ax - \psi(a)y - \dfrac{a}{\psi(a)} = 0$, $\quad x + \psi'(a)y + \dfrac{1}{\psi(a)} - \dfrac{a\psi'(a)}{[\psi(a)]^2} = 0$.

6. C.S. $z = ax + by + \log(ab)$, $\qquad\qquad$ S.S. $\quad z + 2 + \log xy = 0$

   G.S. $z - ax - \psi(a)y - \log a - \log\psi(a) = 0$, $-x - \psi'(a)y - \dfrac{1}{a} - \dfrac{\psi'(a)}{\psi(a)} = 0$.

7. C.S. $z = ax + by + a^2 - b^2$, $\qquad\qquad$ S.S. $\quad x^2 - y^2 + 4z = 0$

   G.S. $z - ax - \psi(a)y - a^2 + [\psi(a)]^2 = 0$, $x + 2a + (y - 2\psi(a))\psi'(a) = 0$

8. C.S. $z = ax + by + a^2 b^2$, $\qquad\qquad$ S.S. $\quad z = -\dfrac{3}{2^{4/3}}x^{2/3}y^{2/3}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 11

G.S. $z - ax - \psi(a)y - a^2\left(\psi(a)\right)^2 = 0$; $-x - \psi'(a)y - 2a\left(\psi(a)\right)^2 - 2a^2\psi(a)\psi'(a) = 0$

9. C.S. $z = ax + by + 3(ab)^{\frac{1}{3}}$,   S.S. $xyz - 1 = 0$

   G.S. $z - ax - \psi(a)y - 3\left(a\psi(a)\right)^{\frac{1}{3}} = 0$; $x + \psi'(a)y + \dfrac{\psi(a) + a\psi'(a)}{\left(a\psi(a)\right)^{\frac{2}{3}}} = 0$

5.4 Type (III) : Equations containing only z, p and q

We consider the differential equation of the form

$$f(p, q, z) = 0 \qquad\qquad ......(1)$$

Charpit's auniliary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + p \cdot f_z} = \frac{dq}{f_y + q f_z}$$

As (1) is free from x and y so $f_x = f_y = 0$ and therefore, we get

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{pf_z} = \frac{dq}{qf_z}$$

Taking the last two fractions, $\dfrac{1}{p}dp = \dfrac{1}{q}dq$

Integrating , $q = ap$,   where a is an arbitrary constant   ......(2)

Now, $dz = p\,dx + q\,dy$

$$= p\,dx + ap\,dy \qquad\qquad [\because q = ap]$$

$$= p\,d(x + ay)$$

$$= p\,du \qquad , \qquad \text{where } u = x + ay$$

$\Rightarrow$ $p = \dfrac{dz}{du}$ and so by (2) , $q = a\dfrac{dz}{du}$

Substituting these values of p and q in (1), we get

$$f\left(\frac{dz}{du},\ a\frac{dz}{du},\ z\right) = 0 \qquad\qquad ......(3)$$

which is an ordinary differential equation of first order. Solving (3), we get z as a function of u. The complete integral in then obtained by replacing u by $x + ay$.

Working rules for solving $f(z, p, q) = 0$

Step (i) : Take $z = G(u)$, where $u = x + ay$

(ii) By putting $p = \dfrac{dz}{du}$ and $q = a\,\dfrac{dz}{du}$, the given equation reduces to an ordinary differential equation of first order. Let its solution be $f(x, y, z, a, b) = 0$. This gives the complete solution of the given equation.

(iii) For singular solution, eliminate a and b from the equations : $f = 0, \dfrac{\partial f}{\partial a} = 0, \dfrac{\partial f}{\partial b} = 0$.

(iv) For general solution, take $b = \phi(a)$ where ϕ is any arbitrary function. The equations : $f = 0, \dfrac{\partial f}{\partial a} = 0$ constitute the general solution .

Example : Find complete integral of $9(p^2 z + q^2) = 4$.

Solution : The given differential equation is $\qquad 9(p^2 z + q^2) = 4 \qquad\qquad$ ......(1)

which is of the form $f(p, q, z) = 0$

Let $u = x + ay$, where a is an arbitrary constant and $z = G(u)$ be solution of (1).

Putting $p = \dfrac{dz}{du}$ and $q = a\left(\dfrac{dz}{du}\right)$ in (1), we get

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 \right] = 4 \quad \text{or} \quad \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z + a^2)}$$

or $\qquad du = \pm \left(\dfrac{3}{2}\right) \times (z + a^2)^{\frac{1}{2}}\, dz$, separating variables u and z

Integrating, $u + b = \pm \left(\dfrac{3}{2}\right) \times \left[\dfrac{(z + a^2)^{\frac{3}{2}}}{\dfrac{3}{2}} \right] \qquad$ or $\quad u + b = \pm\, (z + a^2)^{\frac{3}{2}}$

or $\qquad\qquad (u + b)^2 = (z + a^2)^3 \quad$ or $\quad (x + ay + b)^2 = (z + a^2)^3$, as $u = x + ay$

which is a complete integral containing two arbitrary constants a and b.

Exercise 5.4

Find the complete integral of the following equation :

1. $z = pq$
2. $z = p^2 + q^2$
3. $p(1 - q^2) = q(1 - z)$

4. $z^2 = 1 + p^2 + q^2$
5. $z^2(p^2 z^2 + q^2) = 1$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 13

Solve the following partial differential equation :

6. $p^2 + pq = 4z$ 7. $p^2 + q^2 = 4z$ 8. $z = pq$

9. $4(1 + z^3) = 9z^4 pq$ 10. $p^3 + q^3 = 3pqz$

Answers

1. $4az = (x + ay + b)^2$ 2. $4z(1 + a^2) = (x + ay + c)^2$

3. $4(1 - a + az) = (x + ay + c)^2$ 4. $z = \cosh\left(\dfrac{x + ay + c}{\sqrt{1 + a^2}}\right)$

5. $(z^2 + a^2)^3 = 9(x + ay + c)^2$

6. C.S. $(1 + a)z = (x + ay + c)^2$, S.S. $z = 0$

  G.S. $(1 + a)z - (x + ay + \psi(a))^2 = 0$, $z - 2(x + ay + \psi(a))(y + \psi'(a)) = 0$

7. C.S. $(1 + a^2)z = (x + ay + c)^2$, S.S. $z = 0$

  G.S. $(1 + a^2)z - (x + ay + \psi(a))^2 = 0$, $2az - 2(x + ay + \psi(a))(y + \psi'(a)) = 0$

8. C.S. $4az = (x + ay + c)^2$,   S.S. $z = 0$

  G.S. $4az - (x + ay + \psi(a))^2 = 0$, $2z - (x + ay + \psi(a))(y + \psi'(a)) = 0$

9. C.S. $a(1 + z^3) = (x + ay + c)^2$, S.S. $z^3 + 1 = 0$

  G.S. $a(1 + z^3) - (x + ay + \psi(a))^2 = 0$, $(1 + z^3) - 2(x + ay + \psi(a))(y + \psi'(a))$

10. C.S. $(1 + a^3)\log z = 3a(x + ay) + b$, S.S.   Does not exists

  G.S. $(1 + a^3)\log z - 3a(x + ay) - \psi(a) = 0$, $3a^2 \log z - 3x - 6ay - \psi'(a) = 0$

5.5 Type (IV) Equations of the form $f_1(x, p) = f_2(y, q)$

Type IV : Equations of the form $f_1(x, p) = f_2(y, q)$ i.e., a form in which z is not involved and the

terms containing x and p are on one side and those containing y and q are on the other side.

Let $F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0$ ......(1)

Charpit's auxiliary equation are

$$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{dz}{-pF_p - qF_q} = \frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z}$$

or $\qquad \dfrac{dx}{-\dfrac{\partial f_1}{\partial p}} = \dfrac{dy}{\dfrac{\partial f_2}{\partial q}} = \dfrac{dz}{-p\dfrac{\partial f_1}{\partial p} + q\dfrac{\partial f_2}{\partial q}} = \dfrac{dp}{\dfrac{\partial f_1}{\partial x}} = \dfrac{dq}{-\dfrac{\partial f_2}{\partial y}}$ $\qquad$ [Using (1)]

Taking the first and fourth fractions, we have

$$\frac{\partial f_1}{\partial x}\, dx + \frac{\partial f_1}{\partial p}\, dp = 0$$

$\Rightarrow \qquad\qquad df_1 = 0 \qquad\qquad \Rightarrow \qquad f_1 = a$ where 'a' is an arbitrary constant

So, by (1), we get $\qquad f_1(x,\, p) = a$ and $f_2(y,\, q) = a$

Solving these equations for p and q, we get

$$p = F_1(x,\, a) \quad \text{and} \quad q = F_2(y,\, a)$$

Substituting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = F_1(x,\, a)dx + F_2(y,\, a)dy$$

Integrating both sides, we get

$$z = \int F_1(x,\, a)dx + \int F_2(y,\, a)dy + b \qquad\qquad \text{.......(2)}$$

which is a complete integral of (1).

To find the singular solution : Let $f(x, y, z, a, b) = z - \int F_1(x, a)dx - \int F_2(y, a)dy - b$

$\therefore$ Using $f(x, y, z, a, b) = 0$, $\dfrac{\partial f}{\partial a} = 0$, $\dfrac{\partial f}{\partial b} = 0$, the singular solution is given by elimination a and b

from the equations :

$$z - \int F_1(x, a)dx - \int F_2(y, a)dy - b = 0$$

$$\frac{\partial}{\partial a}\left(\int F_1(x, a)dx + \int F_2(y, a)dy \right) = 0 \quad \text{and} \quad -1 = 0$$

This is impossible , because $-1 \neq 0$

$\therefore$ There is no singular solution

To find the general solution : Let $b = \phi(a)$, where ϕ is an arbitrary function. Using

$f[x, y, z, a, \phi(a)] = 0$, $\dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations :

$$z - \int F_1(x, a)dx - \int F_2(y, a)dy - \phi(a) = 0,\ \frac{\partial}{\partial a}\left(-\int F_1(x, a)dx - \int F_2(y, a)dy \right) - \phi'(a) = 0 \qquad \text{...(3)}$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 15

Working Rule $f_1(x\,,p) = f_2(y\,,q)$:

Steps :

(i)   Take each side of $f_1(x,p) = f_2(y,q)$ equal to a.

(ii)   Solve equations for p and q. Let $p = F_1(x,a), q = F_2(y,a)$

(iii) Equation of the form $f_1(x,p) = f_2(y,q)$ has no singular solution

(iv)   Take the complete solution as $f(x,y,z,a,b) = 0$. Put $b = \phi(a)$. The general solution is given by

the equations : $f\left[x,y,z,a,\phi(a)\right] = 0, \dfrac{\partial f}{\partial a} = 0$.

Example : Find complete integral of $x(1+y)p = y(1+x)q$.

Solution : Separating p and x from q and y, the given equation reduces to

$$\frac{xp}{1+x} = \frac{yq}{1+y}$$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \quad \text{and} \quad \frac{yq}{1+y} = a \qquad \text{so that } p = a\left(\frac{1+x}{x}\right) \text{ and } q = a\left(\frac{1+y}{y}\right)$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}\,dy \text{ or} \qquad dz = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy,$$

Integrating , $z = a(\log x + x) + a(\log y + y) + b = a(\log xy + x + y) + b$,

which is a complete integral containing two arbitrary constants a and b.

Exercise 5.5

Find the complete integral of the following equations :

1. $\sqrt{p} + \sqrt{q} = x + y$ 2. $\sqrt{p} + \sqrt{q} = 2x$ 3. $yp = 2yx + \log q$

4. $p - q = x^2 + y^2$ 5. $x^2 p^2 = q^2 y$ 6. $q = xy\,p^2$

7. $q(p - \cos x) = \cos y$ 8. $\sqrt{p} - \sqrt{q} + 3x = 0$ 9. $x(1+y)\,p = y(1+x)q$

Answers

1. $3z = (x+a)^3 + (y-a)^3 + b$ 2. $6z = (2x+a)^3 + 6a^2 y + b$

3. $az = a^2 x + ax^2 + e^{ay} + b$ 4. $z = \dfrac{1}{3}(x^3 - y^3) + a(x + y) + c$

5. $z = \sqrt{a}\log x + 2\sqrt{ay} + c$ 6. $z = 2\sqrt{ax} + \dfrac{ay^2}{2} + c$

7. $z = ax + \sin x + \dfrac{1}{a}\sin y + c$ 8. $z = -\dfrac{1}{9}(a - 3x)^3 + a^2 y + c$

9. $z = a\log xy + a(x + y) + b$

5.6 General Charpit's Method

General Charpit's Method : Now, we shall take up the general partial differential equations. Some of these problems may fall in the four special types and other may not be of any special type.

Working Rule for using Charpit's method :

(i)   Shift all terms of the given equation to the left side and denote the left side by $f(x, y, z, p, q)$.

(ii)   Find f_x, f_y, f_z, f_p and f_q

(iii)   Write the Charpit's auxiliary equations as $\dfrac{dx}{-f_p} = \dfrac{dy}{-f_q} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z}$

and substitute the values of partial derivatives of f and simplify.

(iv)   Select any two fractions so that the resulting integral is the simplest relation involving at least one of p and q. This relation and the given equation are solved to find the values of p and q.

(v)   Put the values of p and q in the equation $dz = p\,dx + q\,dy$ and integrate. This gives the complete solution of the given equation.

Remark : The solution by Charpit's method is not unique. It depends upon the fractions used from Charpit's auxiliary equations.

Example : Find complete integral of $q = 3p^2$.

Solution : Here given equation is $f(x, y, z, p, q) \equiv 3p^2 - q = 0$ ......(1)

$\therefore$ Charpit's auxiliary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-p f_p - q f_q} = \frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z}$$

or $\dfrac{dx}{-6p} = \dfrac{dy}{1} = \dfrac{dz}{-6p^2 + q} = \dfrac{dp}{0 + p.0} = \dfrac{dq}{0 + q.0}$ [using (1)]   ....(2)

Taking the first fraction of (1) $dp = 0$ so that $p = a$ ……(3)

Substituting this value of p in (1), we get $q = 3a^2$ ……(4)

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 17

$$dz = a\, dx + 3a^2\, dy$$

Integrating both sides, we get $z = ax + 3a^2 y + b$

which is a complete integral, *a* and *b* being arbitrary constants.

Exercise 5.6

Find the complete integral of the following equation using Charpit's method :

1. $z = px + qy + p^2 + q^2$

2. $p^2 - y^2 q = y^2 - x^2$

3. $z^2(p^2 z^2 + q^2) = 1$

4. $p = (z + qy)^2$

5. $2(z + xp + yq) = yp^2$

6. $2z + p^2 + qy + 2y^2 = 0$

7. $(p^2 + q^2)y = qz$

8. $z = px + qy - 2p - 3q$

9. $yzp^2 - q = 0$

10. $(p^2 + q^2)x = pz$

11. $z^2 = 1 + p^2 + q^2$

Answers

The answer of these problems is not unique. However one answer is given.

1. $z = ax + by + a^2 + b^2$

2. $z = \dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right) - \dfrac{a^2}{y} - y + b$

3. $(a^2 z^2 + 1)^3 = 9a^4(ax + y + b)^2$

4. $yz = ax + 2\sqrt{ay} + b$

5. $z = \dfrac{ax}{y^2} - \dfrac{a^2}{4y^3} + \dfrac{b}{y}$

6. $2y^2 z + y^2(a - x)^2 + y^4 = b$

7. $z^2 - a^2 y^2 = (ax + b)^2$

8. $z = ax + by - 2a - 3b$

9. $z^2(a - y^2) = (x + b)^2$

10. $z^2 - a^2 x^2 = (ay + b)^2$

11. $\log\left[z + \sqrt{z^2 - 1}\right] = \dfrac{ax + y}{\sqrt{a^2 + 1}} + c$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

| Page 1 |

Chapter - 6

6.1 Compatible system of first order equations

Consider first order partial differential equations

$$f(x, y, z, p, q) = 0 \qquad \qquad \dots\dots(1)$$

and $$\qquad g(x, y, z, p, q) = 0 \qquad \qquad \dots\dots(2)$$

Equations (1) and (2) are known as compatible if they have atleast one common solution.

To find condition for (1) and (2) to be compatible :

Let $J = $ Jacobian of f and $g \equiv \dfrac{\partial(f, g)}{\partial(p, q)} \neq 0 \qquad \qquad \dots\dots(3)$

Then (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$$p = \phi(x, y, z) \text{ and } q = \psi(x, y, z) \qquad \qquad \dots\dots(4)$$

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition

that the system of equations (4) should be compatibly integrable *i.e.*, the equation

$$dz = p\,dx + q\,dy \quad \text{or} \quad \phi\,dx + \psi\,dy - dz = 0 \qquad \qquad \dots\dots(5) \qquad [\text{using (4)}]$$

should be integrable, (5) is integrable if $\phi\left(\dfrac{\partial\psi}{\partial z} - 0\right) + \psi\left(0 - \dfrac{\partial\phi}{\partial z}\right) + (-1)\left(\dfrac{\partial\phi}{\partial y} - \dfrac{\partial\psi}{\partial x}\right) = 0$

which is equivalent to $\dfrac{\partial\psi}{\partial x} + \phi\dfrac{\partial\psi}{\partial z} = \dfrac{\partial\phi}{\partial y} + \psi\dfrac{\partial\phi}{\partial z} \qquad \qquad \dots\dots(6)$

Substituting from equations (4) in (1) and differentiating w.r.t. '*x*' and '*z*' respectively, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial\psi}{\partial x} = 0 \qquad \qquad \dots\dots(7)$$

and $$\qquad \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z} + \frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z} = 0 \qquad \qquad \dots\dots(8)$$

From (7) and (8), we get $\qquad \dfrac{\partial f}{\partial x} + \phi\dfrac{\partial f}{\partial z} + \dfrac{\partial f}{\partial p}\left(\dfrac{\partial\phi}{\partial x} + \phi\dfrac{\partial\phi}{\partial z}\right) + \dfrac{\partial f}{\partial q}\left(\dfrac{\partial\psi}{\partial x} + \phi\dfrac{\partial\psi}{\partial z}\right) = 0 \qquad \dots\dots(9)$

Similarly (2) yields $\qquad \dfrac{\partial g}{\partial x} + \phi\dfrac{\partial g}{\partial z} + \dfrac{\partial g}{\partial p}\left(\dfrac{\partial\phi}{\partial x} + \phi\dfrac{\partial\phi}{\partial z}\right) + \dfrac{\partial g}{\partial q}\left(\dfrac{\partial\psi}{\partial x} + \phi\dfrac{\partial\psi}{\partial z}\right) = 0 \qquad \dots\dots(10)$

Solving (9) and (10), we get $\dfrac{\partial\psi}{\partial x} + \phi\dfrac{\partial\psi}{\partial z} = \dfrac{1}{J}\left\{\dfrac{\partial(f, g)}{\partial(x, p)} + \phi\dfrac{\partial(f, g)}{\partial(z, p)}\right\} \qquad \qquad \dots\dots(11)$

Again, substituting from equations (4) in (1) and differentiating w.r.t. 'y' and 'z' and proceeding as

before, we obtain $\dfrac{\partial \phi}{\partial y} + \psi \dfrac{\partial \phi}{\partial z} = -\dfrac{I}{J}\left\{\dfrac{\partial(f,g)}{\partial(y,q)} + \psi \dfrac{\partial(f,g)}{\partial(z,q)}\right\}$ ……(12)

substituting from equations (11) and (12) in (1) and replacing ϕ, ψ by p, q respectively, we obtain

$$\dfrac{I}{J}\left\{\dfrac{\partial(f,g)}{\partial(x,p)} + p\dfrac{\partial(f,g)}{\partial(z,p)}\right\} = -\dfrac{I}{J}\left\{\dfrac{\partial(f,g)}{\partial(y,q)} + q\dfrac{\partial(f,g)}{\partial(z,q)}\right\} \text{ or}$$

$$[f,g] = 0 \qquad\qquad ……(13)$$

where $[f,g] \equiv \dfrac{\partial(f,g)}{\partial(x,p)} + p\dfrac{\partial(f,g)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\dfrac{\partial(f,g)}{\partial(z,q)}$

Results :

1. The first order partial differential equations $p = P(x, y)$ and $q = Q(x, y)$ are compatible iff

$$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$$

2. The equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is

homogeneous in x, y, z.

3. The equations $f(x, y, p, q) = 0$, $g(x, y, p, q) = 0$ are compatible if $\dfrac{\partial(f,g)}{\partial(x,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} = 0$

Example 1 : Show that the differential equations $\dfrac{\partial z}{\partial x} = 5x - 7y$ and $\dfrac{\partial z}{\partial y} = 6x + 8y$ are not compatible.

Solution : Given

$$\left.\begin{array}{l}\dfrac{\partial z}{\partial x} = p = 5x - 7y \\[2mm] \dfrac{\partial z}{\partial y} = q = 6x + 8y\end{array}\right\} \qquad ……(1)$$

Comparing (1) with $p = P(x, y)$ and $q = Q(x, y)$, we get

$P = 5x - 7y$ and $Q = 6x + 8y$ ……(2)

We know that $p = P(x, y)$ and $q = Q(x, y)$ are compatible if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

From (2), $\dfrac{\partial P}{\partial y} = -7$ and $\dfrac{\partial Q}{\partial x} = 6$ and so $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$

$\therefore$ System (1) is not compatible.

Example 2 : Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible and solve them.

Solution : Let $f(x, y, z, p, q) = xp - yq = 0$ ……(1)

and $\qquad g(x, y, z, p, q) = z(xp + yq) - 2xy = 0$ ……(2)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

$$\therefore \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial p} \\[2mm] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix} = 2xy, \qquad \frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2 p - xyq,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy, \qquad \frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + y^2 q$$

$$\therefore \left[f,g \right] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= 2xy - x^2 p^2 - xypq - 2xy + xypq + y^2 q^2$$

$$= -xp(xp+yq) + yq(xp+yq)$$

$$= -(xp-yq)(xp+yq) = 0 \qquad\qquad\qquad \text{[Using (1)]}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, we get $p = \dfrac{y}{z}$ and $q = \dfrac{x}{z}$ ……(3)

Using (3) in $dz = pdx + qdy$, we get

$$dz = \frac{y}{z}dx + \frac{x}{z}dy \qquad\qquad \Rightarrow \qquad zdz = ydx + xdy$$

$$\Rightarrow \qquad zdz = d(xy)$$

Integrating both sides, we get

$$\Rightarrow \qquad \frac{z^2}{2} = xy + c$$

$$\Rightarrow \qquad z^2 = 2xy + c \text{, where } c \text{ is an arbitrary constant.}$$
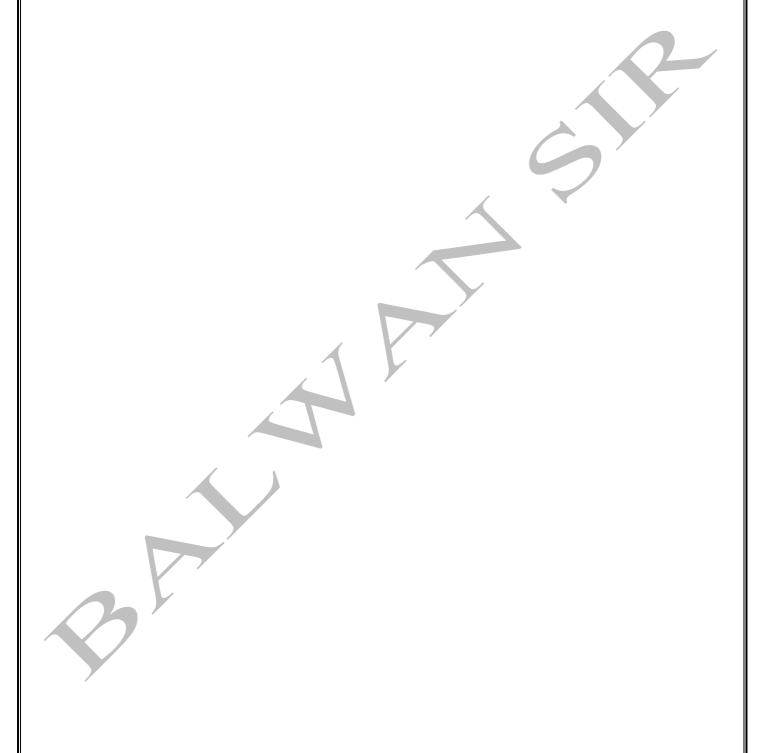
Exercise 6.1

1. Show that the differential equations $p = x^2 - ay, \quad q = y^2 - ax$ are compatible and find their common solution.

2. Show that the differential equations $\dfrac{\partial z}{\partial x} = (x+y)^2, \ \dfrac{\partial z}{\partial y} = x^2 + 2xy - y^2$ are compatible and solve them.

3. Show that the equations $xp - yq = x$ and $x^2 p + q = xz$ are compatible and find their solution.

Answers

1. $z = \dfrac{x^3 + y^3}{3} - axy + c$

2. $z = \dfrac{x^3}{3} + x^2 y + y^2 x - \dfrac{y^3}{3} + c$

3. $z - x = c(1 + xy)$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

---------------- **S C Q** ----------------

1. The complete integral of the partial differential equation

   $xp^3q^2 + yp^2q^3 + (p^3 + q^3) = Zp^2q^2 = 0$ is

   $Z =$

   1. $ax + by + (ab^{-2} + ba^{-2})$

   2. $ax - by + (ab^{-2} - ba^{-2})$

   3. $-ax + by + (ba^{-2} - ab^{-2})$

   4. $ax + by - (ab^{-2} + ba^{-2})$ **(GATE 1997)**

2. Complete integral for the partial differential equation $z = px + qy - \sin(pq)$ is

   1. $z = ax + by + \sin(ab)$

   2. $z = ax + by - \sin(ab)$

   3. $z = ax + y + \sin(b)$

   4. $z = x + by - \sin(a)$ **(GATE 2003)**

3. Let $u = u(x, y)$ be the complete integral of

   the PDE $\dfrac{\partial u}{\partial x} \dfrac{\partial u}{\partial y} = xy$ passing through the

   points $(0,0,1)$ and $\left(0,1,\dfrac{1}{2}\right)$ in the

   $x - y - u$ space. Then the value of $u(x, y)$

   evaluated at $(-1,1)$ is

   1. 0   2. 1

   3. 2   4. 3

   (CSIR NET SCQ Dec 2011)

4. Let $x = x(s)$, $y = y(s)$, $u = u(s)$, $s \in \mathbb{R}$,

   be the characteristic curve of the PDE

   $\left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial u}{\partial y}\right) - u = 0$ passing through a given

   curve $x = 0$, $y = \tau$, $u = \tau^2$, $t \in \mathbb{R}$. Then the

   characteristic are given by

   1. $x = 3\tau(e^s - 1)$, $y = \dfrac{\tau}{2}(e^{-s} + 1)$, $u = \tau^2 e^{-2s}$

   2. $x = 2\tau(e^{-s} - 1)$, $y = \tau(2e^{2s} - 1)$,

      $u = \dfrac{\tau^2}{2}(1 + e^{-2s})$

   3. $x = 2\tau(e^s - 1)$, $y = \dfrac{\tau}{2}(e^s + 1)$, $u = \tau^2 e^{2s}$

   4. $x = \tau(e^{-s} - 1)$, $y = -2\tau\left(e^{-s} - \dfrac{3}{2}\right)$,

      $u = \tau^2(2e^{-2s} - 1)$

      (CSIR NET SCQ June 2014)

5. The Charpit's equations for the PDE

   $up^2 + q^2 + x + y = 0$, $p = \dfrac{\partial u}{\partial x}, q = \dfrac{\partial u}{\partial y}$ are

   given by

   1. $\dfrac{dx}{-1 - p^3} = \dfrac{dy}{-1 - qp^2} = \dfrac{du}{2p^2u + 2q^2} = \dfrac{dp}{2pu} = \dfrac{dq}{2q}$

   2. $\dfrac{dx}{2pu} = \dfrac{dy}{2q} = \dfrac{du}{2p^2u + 2q^2} = \dfrac{dp}{-1 - p^3} = \dfrac{dq}{-1 - qp^2}$

   3. $\dfrac{dx}{up^2} = \dfrac{dy}{q^2} = \dfrac{du}{0} = \dfrac{dp}{x} = \dfrac{dq}{y}$

   4. $\dfrac{dx}{2q} = \dfrac{dy}{2pu} = \dfrac{du}{x + y} = \dfrac{dp}{p^2} = \dfrac{dq}{qp^2}$

      (CSIR NET SCQ Dec 2014)

6. Suppose $u \in C^2(\overline{B})$. B is the unit ball in

   $\mathbb{R}^2$, satisfies $\Delta u = f$ in B, $\alpha u + \dfrac{\partial u}{\partial n} = g$ on

   ∂B, $\alpha > 0$, where a is the unit outward

   normal ot B. If a solution exists then

   1. It is unique
   2. There are exactly two solutions
   3. There are exactly three solutions
   4. There are infinitely many solutions

   (CSIR NET SCQ June 2017)

---------------------- **M C Q** --------------------

1. Given

   $$2xz - \left(\frac{\partial z}{\partial x}\right)x^2 - 2\left(\frac{\partial z}{\partial y}\right)xy + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 0$$

   1. Using Charpit's method, we have

      $$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

   2. Using Charpit's method, we have

      $$\frac{dz}{px^2 + 2xyq - 2pq} = \frac{dp}{2z - 2qy}$$

   3. Complete integral is $z = ay + b(x^2 - a)$

   4. No solution exist

2. Consider the first order PDE $p + q = pq$

   where $p \equiv \dfrac{\partial z}{\partial x}, q \equiv \dfrac{\partial z}{\partial y}$. Then which of the

   following are correct ?

   1. The Charpit's equations for the above
      PDE reduce to

      $$\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{-pq} = \frac{dp}{p+q} = \frac{dq}{0}$$

   2. A solution of the Charpit's equation is

      $q = b$, where b is a constant.

3. The corresponding value of p is

   $$p = \frac{b}{b-1}$$

4. A solution of the equation is

   $$z = \frac{b}{b-1}x + by + a$$, where a and b are

   constants.

   (CSIR NET MCQ June 2013)

3. Let $z = z(x, y)$ be a solution of $\dfrac{\partial z \partial z}{dx\partial y} = 1$

   passing through $(0, 0, 0)$. Then $z(0,1)$ is

   1. 0   2. 1
   3. 2   4. 4

   (CSIR NET MCQ Dec 2013)

4. Let $u = u(x,t)$ be the solution of the

   Cauchy problem

   $$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 1, \quad x \in \mathbb{R}, t > 0$$

   $u(x,0) = -x^2 \quad x \in \mathbb{R}$. Then

   1. $u(x,t)$ exists for all $x \in \mathbb{R}$ and $t > 0$

   2. $|u(x,t)| \to \infty$ as $t \to t^*$ for some $t^* > 0$

      and $x \neq 0$

   3. $u(x,t) \le 0$ for all $x \in \mathbb{R}$ and for all

      $t < \dfrac{1}{4}$

   4. $u(x,t) > 0$ for some $x \in \mathbb{R}$ and

      $0 < t < \dfrac{1}{4}$

   (CSIR NET MCQ Dec 2014)

5. Which of the following are complete

   integrals of the partial differential equation

   $pqx + yq^2 = 1$?

   1. $z = \dfrac{x}{a} + \dfrac{ay}{x} + b$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

2. $z = \dfrac{x}{b} + \dfrac{ay}{x} + b$

3. $z^2 = 4(ax + y) + b$

4. $(z - b)^2 = 4(ax + y)$

(CSIR NET MCQ June 2015)

6. Consider the Cauchy problem for the

    Eikonal equation

    $p^2 + q^2 = 1; \ p \equiv \dfrac{\partial u}{\partial x}, \ q \equiv \dfrac{\partial u}{\partial y} u(x, y) = 0$ on

    $x + y = 1, \ (x, y) \in \mathbb{R}^2$. Then

    1. The Charpit's equations for the

        differential equation are

        $\dfrac{dx}{dt} = 2p; \dfrac{dy}{dt} = 2q; \ \dfrac{du}{dt} = 2;$

    2. The Charpit's equations for the

        differential equation are

        $\dfrac{dx}{dt} = 2p; \dfrac{dy}{dt} = 2q; \ \dfrac{du}{dt} = 2; \dfrac{dp}{dt} = 0; \dfrac{dq}{dt} = 0$

    3. $u(1, \sqrt{2}) = \sqrt{2}$

    4. $u(1, \sqrt{2}) = 1$

    (CSIR NET MCQ June 2016)

7. Consider the Langrange equation
    $x^2 \dfrac{\partial z}{\partial x} + y^2 \dfrac{\partial z}{\partial y} = (x + y)z$. Then the general

    solution of the given equation is

    1. $F\left(\dfrac{xy}{z}, \dfrac{x - y}{z}\right) = 0$ for an arbitrary

        differential function F

    2. $F\left(\dfrac{x - y}{z}, \dfrac{1}{x} - \dfrac{1}{y}\right) = 0$ for an arbitrary

        differentiable function F

    3. $z = f\left(\dfrac{1}{x} - \dfrac{1}{y}\right)$ for an arbitrary

        differentiable function f

4. $z = xy f\left(\dfrac{1}{x} - \dfrac{1}{y}\right)$ for an arbitrary

    differentiable function f

    (CSIR NET Dec 2017)

Answer Key

SCQ

1. 1	2. 2	3. 1
4. 3	5. 2	6. 1

MCQ

1. 1,2,3	2. 2,3,4	3. 2,3,4
4. 2,4	5. 1,4	6. 2,4
7. 1,2,4		

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

Chapter - 7

7.1 Linear Partial Differential Equation of Second And Higher Order

Introduction : A partial differential equation of the first order involves only the first order partial derivatives (*p* and *q*) of the dependent variable *z*. Now we shall proceed the discussion of equation of order higher than one.

In this, we shall use the symbols

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

and operator D for $\dfrac{\partial}{\partial x}$ and D' for $\dfrac{\partial}{\partial y}$.

Linear Partial Differential Equation :

A partial differential equation in which the dependent variable (i.e., *z*) and its partial derivative occur only in the first degree and are not multiplied together, is called a Linear Partial Differential Equation ; otherwise it is called a Non linear Partial differential equation.

For example, $x^2 \dfrac{\partial^2 z}{\partial x^2} - \dfrac{3\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} - y \dfrac{\partial z}{\partial y} = 0$ is a linear partial differential equation.

whereas $xz \dfrac{\partial z}{\partial x} + yz \dfrac{\partial z}{\partial y} = xy$ is a non linear partial differential equation.

Order of a Partial Differential Equation :

By order of a partial differential equation, we mean the order of the highest partial derivative occurring in the given partial differential equation

For example , $\dfrac{\partial^3 z}{\partial x^3} + 2 \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial y} = x^2 + y$ is a partial differential equation of order 3.

General Linear Partial Differential Equation of order *n* : General linear partial differential equation of order *n* is

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + ... + A_n \frac{\partial^n z}{\partial y^n} \right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + ... + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} \right)$$

$$+ ...... + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y} \right) + N_0 z = f(x, y) \qquad ......(1)$$

where the coefficients $A_0, A_1, \ldots, A_n, B_0, B_1, \ldots B_{n-1}, \ldots, M_0, M_1$ and N_0 are constants or function of x and y. If these coefficients are all constants then such a differential equation is called a Linear Partial Differential Equation with Constant Coefficients.

Equation (1) can be written in symbolic form as

$$\Big[(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \ldots + A_n D'^n) + (B_0 D^{n-1} + B_1 D^{n-2} D' + \ldots + B_{n-1} D'^{n-1}) + \ldots$$

$$\ldots + (M_0 D + M_1 D') + N_0 z = f(x, y) \qquad \text{where } D \equiv \frac{\partial}{\partial x} \text{ and } D' \equiv \frac{\partial}{\partial y} \Big]$$

Linear Homogeneous Partial Differential of order n : A linear partial differential equation in which the order of all partial derivatives is same is called the homogeneous linear partial differential equation. Therefore , a homogeneous linear partial differential equation of order n will be of the form

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \ldots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y)$$

If all the partial derivatives are not of same order, then the equation is called a non homogeneous partial differential equation.

Solution of Homogeneous Partial Differential Equation with constant coefficients : Solution of partial differential equation consists two parts complementary function (C.F.) and particular integral (P.I.).

If we have partial differential equation

$$F(D, D')z = f(x, y) \qquad \ldots (1)$$

Then complementary function is the solution of $F(D, D') \, z = 0$ and it contains as many arbitrary constants as is the order of partial differential equation and any particular solution of (1) which contain no arbitrary constant is called a Particular Integral of (1).

The general solution of (1) is given by

$$z = C.F. + P.I.$$

Theorem 1 : Let $F(D, D')z = f(x, y)$ be a linear partial differential equation with constant coefficients. If u is solution of $F(D, D')z = 0$ i.e., u is complementary function, and v is a solution of $F(D, D') = f(x, y)$ (i.e., v is P.I.) , then $u + v$ is a solution of $F(D, D')z = f(x, y)$.

Theorem 2 : If $u_1, u_2, \ldots u_n$ are solutions of the homogeneous linear partial differential equation $F(D, D')z = 0$, then their linear combination $\sum\limits_{r=1}^{n} m_r u_r$ is also a solution of $F(D, D')z = 0$, where $m_1, m_2, \ldots m_n$ are arbitrary constants.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

Working Rule :

Step 1 : To find the general solution of the equation $F(D, D')z = 0$ or to find the C.F. of

$F(D, D')z = f(x, y)$, put $D = m$ and $D' = 1$ to get Auxiliary equation in variable m .

We solve the equation for m. Two cases will arise.

Case 1 : If $m = m_1, m_2..........m_n$ then

$$\text{C.F.} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + ...\phi_n(y + m_n x)$$

where $\phi_1, \phi_2............\phi_n$ are arbitrary functions

and if $\qquad\qquad m = \dfrac{a_1}{b_1} , \ \dfrac{a_2}{b_2}, ...\dfrac{a_n}{b_n}$

then $\qquad\qquad \text{C.F.} = \phi_1(b_1 y + a_1 x) + \phi_2(b_2 y + a_2 x) + ...\phi_n(b_n y + a_n x)$

Case 2 : Let $m = m'$ repeated r times and $m_{r+1}, m_{r+2}...........m_n$ are different roots , then

$$\text{C.F.} = \phi_1(y + m'x) + x\phi_2(y + m'x) + ...x^{r-1}\phi_r(y + m'x) + \phi_{r+1}(y + m_{r+1}x)$$

$$+\phi_{r+2}(y + m_{r+2}x) + ...\phi_n(y + m_n x)$$

Case 3 : Corresponding to a non-repeated factor D on L.H.S. of equation

$$(A_0 D^n + A_1 D^{n-1} D' + ...A_n D'^{\,n}) z = f(x, y) \qquad\qquad ......(1)$$

The C.F. is $\phi(y)$

Case 4 : Corresponding to a repeated factor D^m on L.H.S. of (1), the C.F. is

$$\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + ...x^{m-1}\phi_m(y)$$

Case 5 : Corresponding to a non-repeated factor D' on L.H.S. of (1), the C.F. is $\phi(x)$

Case 6 : Corresponding to a repeated factor D'^m on L.H.S. of (1) , the C.F. is taken as

$$\phi_1(x) + y\ \phi_2(x) + y^2\ \phi_3(x) + ...y^{m-1}\phi_m(x)$$

Alternative working rule for finding C.F. :

We have the partial differential equation $F(D, D')z = f(x, y)$

Factorize $F(D, D')$ into linear factor of the form $(bD - aD')$ then

Case 1 : Corresponding to each non-repeated factor $(bD - aD')$, the C.F. is taken as $\phi(by + ax)$

Case 2 : Corresponding to repeated factor $(bD - aD')^m$. we will take C.F. as

$$\phi_1(by + ax) + x\phi_2(by + ax) + ...x^{m-1}\phi_m(by + ax)$$

Case 3 : For non-repeated factor D. The C.F. is $\phi(y)$

Case 4 : For repeated factor D^m the C.F. is $\phi_1(y) + x\,\phi_2(y) + \ldots x^{m-1}\,\phi_m(y)$

Case 5 : Corresponding to a non-repeated factor D' , the C.F. is $\phi(x)$

Case 6 : For repeated factor $(D')^m$ the C.F. is

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \ldots\ldots y^{m-1}\phi_m(x)$$

Example : Solve $2r + 5s + 2t = 0$

Solution : As we know that

$$r = \frac{\partial^2 z}{\partial x^2}\,, \quad s = \frac{\partial^2 z}{\partial x\,\partial y}\,, \quad t = \frac{\partial^2 z}{\partial y^2}$$

$\therefore$ equation (1) can be written as

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

Its symbolic form is

$$(2D^2 + 5DD' + 2D'^2)\,z = 0 \quad \left[\text{Using } \frac{\partial}{\partial x} = D,\; \frac{\partial}{\partial y} = D'\right]$$

Its *A.E.* is $\qquad 2D^2 + 5DD' + 2D'^2 = 0$

Putting $D = m$, $D' = 1$, we get $\qquad 2m^2 + 5m + 2 = 0$

$$2m^2 + 4m + m + 2 = 0$$

or $\qquad 2m(m+2) + 1(m+2) = 0$

or $\qquad (2m+1)(m+2) = 0$

$\Rightarrow \qquad m = -\dfrac{1}{2},\; m = -2$

so the general solution of equation (1) is

$$z = \phi_1\!\left(y - \frac{1}{2}x\right) + \phi_2(y - 2x)$$

$$= \psi_1(2y - x) + \phi_2(y - 2x)$$

where ψ_1 , ϕ_2 are arbitrary functions.

Exercise 7.1

Find the general solution of the following partial differential equations :

1. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x\,\partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = 0$
2. $(D^3 - 3DD'^2 - 2D'^3)z = 0$

3. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^3 z}{\partial y^3} = 0$
4. $(D^2 D' - 2DD'^2 + D'^3)z = 0$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

5. $(D'^3 D + D'^4) z = 0$ 　　　　　　　　　　　　6. $(D^2 - D'^2 + D.D') z = 0$

7. $\dfrac{\partial^3 z}{\partial x^3} - 4 \dfrac{\partial^3 z}{\partial x^2 \partial y} + 4 \dfrac{\partial^3 z}{\partial x \partial y^2} = 0$

Answers

1. $z = \phi_1(y - x) + \phi_2(y + 2x)$

2. $z = \phi_1(y + 2x) + \phi_2(y - x) + x\,\phi_3(y - x)$

3. $z = \phi_1(y + x) + \phi_2(y + wx) + \phi_3(y + w^2 x)$　 where w, w^2 are cube root of unity.

4. $z = \phi_1(x) + \phi_2(y + x) + x\,\phi_3(y + x)$

5. $z = \phi_1(x) + y\,\phi_2(x) + y^2\,\phi_3(x) + \phi_4(y - x)$

6. $z = \phi_1(y + \alpha x) + \phi_2(y + \beta x)$　　　where $\alpha = \dfrac{-1 + \sqrt{5}}{2}$, $\beta = \dfrac{-1 - \sqrt{5}}{2}$

7. $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$

7.2 Particular Integral

Particular Integral : Particular Integral of $F(D, D') z = f(x, y)$ 　　　　　　　　......(1)

The inverse operator $\dfrac{1}{F(D, D')}$ of the operator $F(D, D')$ is defined by the following identity

$$F(D, D')\left(\frac{1}{F(D, D')} f(x, y) \right) = f(x, y)$$

Thus , the particular integral of (1) is $\dfrac{1}{F(D, D')} f(x, y)$.

Remarks :

(i) We will use D, D^2, D^3........ as partial differentiation with respect to x once, twice, thrice and so on.

i.e., 　　　　　　　　　　$D = \dfrac{\partial}{\partial x}$, $D^2 = \dfrac{\partial^2}{\partial x^2}$, $D^3 = \dfrac{\partial^3}{\partial x^3}$, ...

For example , 　　　　　　$D^2 x^2 y^4 = D(2x\, y^4) = 2y^4$

(ii) We will use D', D'^2, D'^3 as differentiating partially w.r.t. y once, twice, thrice and so on i.e.,

$$D' = \frac{\partial}{\partial y}, \quad D'^2 = \frac{\partial^2}{\partial y^2}, \quad D'^3 = \frac{\partial^3}{\partial y^3}, \dots$$

(iii) $\frac{1}{D}$ stands for integrating partially w.r.t. x i.e., $\frac{1}{D} x^2 y^3 = \int x^2 y^3 \, dx = \frac{x^3}{3} y^3$

$\frac{1}{D'}$ stands for integrating partially w.r.t. y i.e., $\frac{1}{D'} x^2 y^3 = \int x^2 y^3 \, dy = \frac{x^2 y^4}{4}$

Different Methods for finding Particular Integrals :

Consider the equation

$$(D^2 + mDD' + nD'^2) z = f(x, y)$$

We can write it as $\qquad F(D, D') z = f(x, y)$

Case 1 : To find *P.I.* when $f(x, y) = e^{ax+by}$

$$P.I. = \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$$

But this result fail when $F(a, b) = 0$. In this situation, we will adopt another method which will be discussed later on.

Case 2 : To find *P.I.* of $f(D, D') = f(x, y)$ when $f(x, y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \frac{1}{F(D, D')} \sin(ax+by) = \frac{1}{D^2 + mDD' - nD'^2} \sin(ax+by)$$

$$= \frac{1}{-a^2 - mab - nb^2} \sin(ax+by)$$

i.e., We will substitute $\qquad \begin{array}{ccc} D^2 & \text{to} & -a^2 \\ DD' & \text{to} & -ab \\ \text{and} \quad D'^2 & \text{to} & -b^2 \end{array}$ in $\dfrac{1}{F(D, D')} \sin(ax+by)$

But this result fails when $-a^2 - mab - nb^2 = 0$, to handle these types of problems, we will discuss a method later on.

Similarly, we can find P.I. , when $f(x, y) = \cos(ax+by)$

General Method of finding the particular integrals :

$$\frac{1}{(D - mD')} f(x, y) = \int f(x, c - mx) \, dx \qquad , \qquad \text{where } c = y + mx$$

Since *P.I.* does not contain any arbitrary constants

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 7

After Integration of (5) constant *c* must be replaced by *y* + *mx*

∴ The *P.I.* given by (2) can be obtained by applying the operation (5) by the factors term by term starting from right.

A Shorter method to find the *P.I.* , when $f(x, y)$ is of the form $f(ax+by)$

It should be noted that all the problems which we will do in case 1, 2 can be done with this shorter method including case of failure of case (1) and (2). We will do this method in the form of these two theorem.

Theorem 1 : If $F(D, D')$ be Homogeneous function of *D* and *D'* of degree *n* then

$$\frac{1}{F(D,D')}\phi^n(ax+by)=\frac{1}{F(a,b)}\phi(ax+by)$$ provided $F(a,b)\neq 0$, $\phi^n(ax+by)$ means the *n*th derivative

of ϕ w.r.t. *ax* + *by* as a whole.

Theorem 2 : $\dfrac{1}{(bD-aD')^n}\phi(ax+by)=\dfrac{x^n}{n!b^n}\phi(ax+by)$.

Shorter Method :

(1) $\dfrac{1}{F(D,D')}\phi(ax+by)=\dfrac{1}{F(a,b)}\iint............\int\phi(v)\,dv\,dv.......dv$

  where $v = ax + by$ provided $F(a, b) \neq 0$

(2)   when $F(a,b)=0$

  Then $\dfrac{1}{(bD-aD')^r}\phi(ax+by)=\dfrac{x^r}{b^r\,r!}\phi(ax+by)$

We can do all problems of case 1 and case 2 by this shorter method and in the situation of case of failure of case 1 and case 2, we can use 2nd result of shorter method. It is better to do the problems of case (2) by shorter Method directly.

Example 1 : Solve $(D^2+2DD'+D'^2)\,z=e^{2x+3y}$

Solution : Given equation is

$$(D^2+2DD'+D'^2)\,z=e^{2x+3y}$$

Its Auxiliary equation is

$$m^2+2m+1=0$$

By putting $D=m$, $D'=1$, we get

$$(m+1)^2 = 0 \implies m = -1, -1$$

∴ Complementary function is

$$\phi_1(y - x) + x\phi_2(y - x)$$

Now for particular integral

$$\frac{1}{F(D, D')} e^{2x+3y} = \frac{1}{D^2 + 2DD' + D'^2} e^{2x+3y}$$

Put $D = 2$, $D' = 3$, $DD' = 6$

$$= \frac{1}{4 + 12 + 9} e^{2x+3y} = \frac{e^{2x+3y}}{25}$$

∴ General solution is

$$z = C.F. + P.I.$$

$$= \phi_1(y - x) + x\,\phi_2(y - x) + \frac{e^{2x+3y}}{25}$$

Example 2 : Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$.

Solution : Auxiliary equation of the given equation is

$$m^2 - m - 2 = 0$$

$$(m - 2)(m + 1) = 0, \ m = 2, -1$$

C.F. is $\qquad z = \phi_1(y + 2x) + \phi_2(y - x)$

Now for *P.I.* is $\qquad \dfrac{1}{D^2 - DD' - 2D'^2}(2x^2 + xy - y^2)\sin xy - \cos xy$

$$= \frac{1}{(D - 2D')(D + D')}[(2x - y)(x + y)]\sin xy - \cos xy \quad \begin{bmatrix} 2x^2 + xy - y^2 = \\ 2x^2 + 2xy - xy - y^2 \\ = 2x(x + y) - y(x + y) \\ = (2x - y)(x + y) \end{bmatrix}$$

$$= \frac{1}{(D - 2D')}\left[\frac{1}{(D + D')}(2x - y)(x + y)\sin xy - \frac{1}{(D + D')}\cos xy \right]$$

Using General Method

$$= \frac{1}{(D - 2D')}\left[\frac{1}{(D + D')}(2x - y)(x + y)\sin xy - \frac{1}{D + D'}\cos xy \right]$$

$$= \frac{1}{D - 2D'}\left[\int (2x - c - x)(x + c + x)\sin x(c + x)\,dx - \int \cos x(c + x)\,dx \right]$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 9

[Integration by part of 1st integration by taking

$(2x+c)\sin(cx+x^2)$ as 2nd function]

$$= \frac{1}{(D-2D')}\Big[-(x-c)\cos(cx+x^2)+\int\cos(cx+x^2)dx-\int\cos(cx+x^2)\,dx\Big]$$

$$\left[\begin{array}{l} \because \text{ Since } \dfrac{d}{dx}\cos(cx+x^2)=-\sin(cx+x^2)(2x+c) \\ \therefore \int\sin(cx+x^2)(2x+c)=-\cos(cx+x^2) \end{array}\right]$$

$$= \frac{1}{D-2D'}\Big[(-x+y-x)\cos(cx+x^2)\Big]$$

$$= \frac{1}{(D-2D')}(y-2x)\cos(cx+x^2)$$

$$= \frac{1}{(D-2D')}(y-2x)\cos xy$$

Now , $m=2,\ y+mx=c,\ y+2x=c,\ y=c-2x$

Again $\int(c-2x-2x)\cos x(c-2x)\,dx=\int(c-4x)\cos(cx-2x^2)dx$

Put $cx-2x^2=t$, $(c-4x)\,dx=dt$

$$\int\cos t\ dt=+\sin t=\sin(cx+2x^2)$$

$$P.I. = \sin x(c-2x)=\sin xy$$

Hence the complete solution is

$$z=\phi_1(y-x)+\phi_2(y+2x)+\sin xy$$

Example 3 : Solve $(D^2-4D'^2)z=\left(4\dfrac{x}{y^2}-\dfrac{y}{x^2}\right)$.

Solution : Auxiliary equation of the given equation is

$$m^2-4=0$$

$$m=\pm 2$$

$\therefore$ Complementary function is $z=\phi_1(y+2x)+\phi_2(y-2x)$

$$P.I. = \frac{1}{D^2-4D'^2}\left(4\,\frac{x}{y^2}-\frac{y}{x^2}\right)$$

$$= \frac{1}{(D+2D')(D-2D')}\left[\frac{4x}{y^2} - \frac{y}{x^2}\right]$$

$$= \left(\frac{1}{D+2D'}\right)\left[\frac{1}{(D-2D')}\left(\frac{4x}{y^2} - \frac{y}{x^2}\right)\right]$$

$$= \frac{1}{(D+2D')}\left[\int\left(\frac{4x}{(c-2x)^2} - \frac{c-2x}{x^2}\right)\right]dx \qquad \begin{bmatrix} \text{Using General Method for } P.I. \\ \dfrac{1}{D-mD'}f(x,y) = \displaystyle\int f(x,c-mx)\,dx \\ \text{Here } m=2,\ y+2x=c,\ y=c-2x \end{bmatrix}$$

$$= \frac{1}{(D+2D')}\int\left(\frac{-2(c-2x)+2c}{(c-2x)^2} - \frac{c}{x^2} + \frac{2}{x}\right)dx$$

$$= \frac{1}{(D+2D')}\left[\int\left(\frac{-2}{c-2x} + \frac{2c}{(c-2x)^2} - \frac{c}{x^2} + \frac{2}{x}\right)dx\right]$$

$$= \frac{1}{(D+2D')}\left[\frac{-2\log(c-2x)}{-2} + \frac{-2c}{(c-2x)\,(-2)} + \frac{c.1}{x} + 2\log x\right]$$

$$= \frac{1}{(D+2D')}\left[\log(c-2x) + \frac{c}{c-2x} + \frac{c}{x} + 2\log x\right]$$

$$= \frac{1}{(D+2D')}\left[\log(y+2x-2x) + \frac{y+2x}{y+2x-2x} + \frac{y+2x}{x} + 2\log x\right]$$

$$= \frac{1}{D+2D'}\left[\log y + \frac{y+2x}{y} + \frac{y+2x}{x} + 2\log x\right]$$

Again using General Method of P.I.

$$\int\left[\log(c+2x) + \frac{c+2x}{c+2x} + \frac{2x}{c+2x} + \frac{c+4x}{x} + 2\log x\right]dx$$

$$\text{Here } m=-2,\ y-2x=c,\ y=c+2x$$

$$= \int\log(c+2x)\,dx + \int 1\,dx + \int\frac{2x}{c+2x}\,dx + \int\frac{c}{x}\,dx + \int 4\,dx + \int 2\log x\,dx$$

$$= \log(c+2x)\,x - \int\frac{2x}{c+2x}\,dx + x + \int\frac{2x}{c+2x}\,dx + c\log x + 4x + 2[\log x.x - \int\frac{1}{x}\cdot x\,dx]$$

$$= x\log(c+2x) + 5x + c\log x + 2x\log x - 2x$$

$$= x\log(c+2x) + 3x + c\log x + 2x\log x$$

Put $c = y - 2x$

$$= x\log(y-2x+2x) + 3x + (y-2x)\log x + 2x\log x$$

$$= x\log y + 3x + y\log x$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 11

∴ The complete solution is

$$z = \phi_1(y+2x) + \phi_2(y-2x) + x\log y + 3x + y\log x$$

Exercise 7.2

Solve the following partial differential equations :

1. (a) $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$ (b) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}$

   (c) $2r - s - 3t = \dfrac{5e^x}{e^y}$ (d) $(D^3 - 7DD'^2 - 6D'^3)z = e^{3x+y}$

2. (a) $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)$

   (b) $(D^2 - 5DD' + 4D'^2)z = \cos(4x+y)$

   (c) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+y)$

   (d) $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x+2y)$

   (e) $(D^3 - 4D^2D' + 4D(D')^2 = 2\sin(3x+2y)$

3. (a) $(D^2 - DD' - 2D'^2)z = 2x+3y$ (b) $(2D^2 - 5DD' + 2D'^2)z = 5(y-x)$

   (c) $(D^2 - 2DD' + D'^2)z = \tan(y+x)$ (d) $(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$

4. (a) $(D^2 - DD' - 2D'^2)z = (y-1)e^x$ (b) $r + s - 6t = y\cos x$

5. $(D^2 + DD' - 6D'^2)z = x^2\sin(x+y)$

6. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x\cos 2y$

7. $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$

8. $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x+2y)$

9. $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{\frac{1}{2}}$

10. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x\partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x$ 11. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \cos 2x \cos 3y$

12. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = (y+2)e^x$ 13. $D^2 - 4DD' + 3D'^2 = \sqrt{x+3y}$

Answers

1. (a) $z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y-x) - \dfrac{x}{2}e^{x+y}$

(b) $z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x) - \dfrac{1}{91} e^{5x + 6y}$

(c) $z = \phi_1(y - x) + \phi_2(2y + 3x) + x e^{x - y}$

(d) $z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) + \dfrac{x\, e^{3x + y}}{20}$

2.   (a) $z = \phi_1(y + 2x) + \phi_2(2y + x) - \dfrac{5x}{3} \cos(2x + y)$

(b) $z = \phi_1(y + x) + \phi_2(y + 4x) + \dfrac{x}{3} \sin(4x + y)$

(c) $z = \phi_1(y) + \phi_2(y + 2x) + x\, \phi_3(y + 2x) + \dfrac{x^2}{4} \sin(2x + y)$

(d) $z = \phi_1(y + 2x) + \phi_2(y - x) + x\, \phi_3(y - x) + \dfrac{1}{27} \sin(x + 2y)$

(e) $z = \phi_1(y) + \phi_2(y + 2x) + x \phi_3(y + 2x) + \dfrac{2}{3} \cos(3x + 2y)$

3.   (a) $z = \phi_1(y + 2x) + \phi_2(y - x) - \dfrac{(2x + 3y)^3}{120}$ (b) $z = \phi_1(2y + x) + \phi_2(y + 2x) + \dfrac{5}{54}(y - x)^3$

(c) $z = \phi_1(y + x) + x\, \phi_2(y + x) + \dfrac{x^2}{2!} \tan(y + x)$

(d) $z = \phi_1\left(y + \dfrac{1}{2}x\right) + x\, \phi_2\left(y + \dfrac{x}{2}\right) + 2x^2 \log(2y + x)$

4.   (a) $z = \phi_1(y - x) + \phi_2(y + 2x) + y e^x$ (b) $z = \phi_1(y - 3x) + \phi_2(y + 2x) - y \cos x + \sin x$

5. $z = \phi_1(y - 3x) + \phi_2(y + 2x) + \left(\dfrac{x^2}{4} - \dfrac{13}{32}\right) \sin(x + y) - \dfrac{3}{8} x \cos(x + y)$

6. $z = \phi_1(y + x) + \phi_2(y - x) + x\, \phi_3(y - x) + \dfrac{e^x}{25}(\cos 2y + 2 \sin 2y)$

7. $z = \phi_1(y - x) + x\, \phi_2(y - x) + x \sin y$

8. $z = \phi_1(y + x) + \phi_2(y + 2x) + \dfrac{1}{12} e^{2x - y} - x e^{x + y} - \dfrac{1}{3} \cos(x + 2y)$

9. $z = \phi_1(y + 2x) + \phi_2(y + x) + x\, \phi_3(y + x) + x e^{y + 2x} - \dfrac{2}{3} \dfrac{x^2}{2!} (y + x)^{\frac{3}{2}}$

10. $z = \phi_1(y + x) + x\, \phi_2(y + x) - \sin x$ 11. $z = \phi_1(y + ix) + \phi_2(y - ix) - \dfrac{1}{13} \cos 2x \cos 3y$

12. $z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x) + y e^x$ 13. $z = \phi_1(y + x) + \phi_2(y + 3x) + \dfrac{1}{60}(3y + x)^{\frac{5}{2}}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 13

7.3 Method to find *P.I.* when $f(x, y)$ is of the form $x^m y^n$, where *m* and *n* are integers

$$P.I. = \frac{1}{F(D,D')} x^m y^n = F(D, D')^{-1} x^m y^n$$

We will expand $F(D, D')^{-1}$ with the help of binomial expansion either in ascending power of *D* or *D'*

and operate on $x^m y^n$ term by term. It should be noted that *P.I.* obtained by expanding $\frac{1}{f(D,D')}$ in

ascending power of *D* is different from that obtained on expanding $\frac{1}{f(D,D')}$ in ascending power of

D. We can use any one of them.

Remark : If $n < m$ then expend $\frac{1}{f(D,D')}$ in powers of $\frac{D'}{D}$ and when $m < n$, expand $\frac{1}{f(D,D')}$ in

powers of $\frac{D}{D'}$

Example 1 : Solve $(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3 + \cos(x - y)$.

Solution : A.E. of given equation is $m^3 - 7m - 6 = 0$ or $(m+1)(m+2)(m-3)=0$

$\therefore$ $\qquad\qquad\qquad\qquad\qquad m = -1, -2, 3$

$\therefore$ $\qquad\qquad\quad C.F. = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary function

$\quad$ *P.I.* Corresponding to $x^2 + xy^2 + y^3$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3\left[1 - 7\left(\dfrac{D'}{D}\right)^2 - 6\left(\dfrac{D'}{D}\right)^3\right]}(x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}\left[1 - \left(7\left(\dfrac{D'}{D}\right)^2 + 6\left(\dfrac{D'}{D}\right)^3\right)\right]^{-1}(x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}\left[1 + 7\left(\dfrac{D'}{D}\right)^2 + 6\left(\dfrac{D'}{D}\right)^3 + ...\right](x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}\left[(x^2 + xy^2 + y^3) + \frac{7}{D^2}(2x + 6y) + \frac{6}{D^3}6 + 0 + 0 + ...\right]$$

$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{7}{D^5}(2x + 6y) + \frac{36.1}{D^6}$$

Now $D'[D'(x^2 + xy^2 + y^3)] = D'\left(2xy + 3y^2\right) = 2x + 6y, \ D'^3(x^2 + xy + y^3) = 6$

$$= \frac{x^5}{3.4.5} + \frac{x^4 y^2}{2.3.4} + \frac{y^3 x^3}{2.3} + \frac{2x7.x^6}{2.3.4.5.6} + \frac{42.y x^5}{2.3.4.5} + 36\frac{x^6}{2.3.4.5.6}$$

$$= \frac{x^5}{60} + \frac{x^4 y^2}{24} + \frac{x^3 y^3}{6} + \frac{x^6}{360} + \frac{7.x^5 y}{20} + \frac{x^6}{20} = \frac{x^5}{60} + \frac{5x^6}{72} + \frac{x^3 y^3}{6} + \frac{x^4 y^2}{24} + \frac{7}{20}x^5 y$$

Now, *P.I.* corresponding to cos $(x - y)$

$$\frac{1}{D^3 - 7D.D'^2 - 6D'^3}\cos(x-y) = \frac{1}{(D+D')(D^2 - DD' - 6D'^2)}\cos(x-y)$$

Put $D^2 = -1, \ DD' = (-1)(-1), \ D'^2 = -1$

$$= \frac{1}{(D+D')}\left[\frac{1}{-1-1+6}\cos(x-y)\right] = \frac{1}{4(D+D')}\cos(x-y)$$

since $F(a, b) = 0$

$\therefore$ By using 2nd result of shorter method

$$= -\frac{1}{4}\frac{x}{1!(-1)}\cos(x-y) = \frac{x}{4}\cos(x-y) \ , \text{ Here } b = -1, a = 1$$

$\therefore$ The general solution is

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) + \frac{x^5}{60} + \frac{5}{72}x^6 + \frac{7}{20}x^5 y + \frac{1}{24}x^4 y^2 + \frac{1}{6}x^3 y^3 + \frac{x}{4}\cos(x-y)$$

Example 2 : Solve $(D^2 D' - 2DD'^2 + D'^3)z = \frac{1}{x^2}$.

Solution : Given equation is $(D^2 D' - 2DD'^2 + D'^3)z = \frac{1}{x^2}$ or $D'(D-D')^2 = \frac{1}{x^2}$ ......(1)

To find *C.F.* we will use Alternative working Rule because *A. E.* of (1) will give only two values But $F(D,D')$ is in 3rd degree

$\therefore$ *C.F.* is $\phi_1(x) + \phi_2(y+x) + x\ \phi_3(y+x)$

Now $P.I. = \frac{1}{(D-D')^2 D'}\frac{1}{x^2} = \frac{1}{(D-D')^2}\left(\frac{y}{x^2}\right)$ as $\frac{1}{D'}\left(\frac{1}{x^2}\right) = \frac{y}{x^2}$

$$= \frac{1}{D^2}\frac{1}{\left[\left(1 - \left(\frac{D'}{D}\right)\right)\right]^2}\frac{y}{x^2} = \frac{1}{D^2}\left[1 - \left(\frac{D'}{D}\right)\right]^{-2}\left(\frac{y}{x^2}\right)$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 15

$$= \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + \frac{2.3.}{2}\left(\frac{D'}{D}\right)^2 + \dots\dots\right]\left(\frac{y}{x^2}\right) \qquad [\because D'^2 y = 0]$$

$$= \frac{1}{D^2}\left[\frac{y}{x^2} + \frac{2}{D}\left(\frac{1}{x^2}\right) + 0 + 0\dots\dots\right]$$

$$= \frac{1}{D^2}\left[\frac{y}{x^2}\right] + \frac{2}{D^3}\left(\frac{1}{x^2}\right)$$

$$= y\left[\frac{1}{D}\left(-\frac{1}{x}\right) + \frac{2}{D^2}\left(-\frac{1}{x}\right)\right]$$

$$= -y\log x + \frac{2}{D}[-\log x]$$

$$= -y\log x - 2\left[\log x.\ x - \int \frac{1}{x}\ x\ dy\right] = -y\log x - 2x\log x + 2x$$

$\because\ -2x\log x + 2x$ are any function of x and we have in C.F. a function $\phi(x)$

$\therefore$ There two included in $\phi(x)$ No need to write them separately in general solution.

$\therefore \qquad\qquad z = \phi_1(x) + \phi_2(y+x) + x\ \phi_3(y+x) - y\log x$

Exercise 7.3

Solve the following partial differential equation :

1. $(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$

2. $(2D^2 - 5DD' + 2D'^2)z = 24(y-x)$

3. $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$

4. $r + (a+b)s + abt = xy$

5. $(D^2 - 2DD' + D'^2)z = 12xy$

Answers

1. $z = \phi_1(y-x) + x\phi_2(y-x) + \frac{1}{4}(x^4 - 2x^3 y + 2x^2 y^2)$

2. $z = \phi_1(y+2x) + \phi_2(2y+x) + 6x^2 y + 3x^3$

3. $z = \phi_1(y + x) + x\, \phi_2(y + x) + e^{x+2y} + \dfrac{x^5}{20}$

4. $z = \phi_1(y - ax) + \phi_2(y - bx) + \dfrac{x^3 y}{6} - \dfrac{(a+b)x^4}{24}$

5. $z = \phi_1(y + x) + x\phi_2(y + x) + 2x^3 y + x^4$

3. $z = \phi_1(y + x) + x\, \phi_2(y + x) + e^{x+2y} + \dfrac{x^5}{20}$

4. $z = \phi_1(y - ax) + \phi_2(y - bx) + \dfrac{x^3 y}{6} - \dfrac{(a+b)x^4}{24}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

Chapter - 8

8.1 Non-Homogenous Linear Partial Differential Equations with Constant Coefficients

Def. A linear partial differential equation with constant coefficients is called non homogenous, if the order of all the partial derivatives involved in the equation are not equal.

For example $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^3 z}{\partial y^3} + \dfrac{\partial z}{\partial x} + z = x + y$

Def. Reducible and Irreducible linear differential operator : A linear differential operator $F(D, D')$ is known as reducible, if it can be written as the product of linear factors of the form $aD + bD' + c$, where a, b, c are constants , other wise it is said to be irreducible.

For example $D^2 - D'^2 = (D + D')(D - D')$ is reducible and $D^2 - D'^3$ is irreducible as we cannot factorize it.

Def. Reducible and Irreducible linear differential equation : A linear partial differential equation $F(D, D')z = f(x, y)$ is reducible if $F(D, D')$ is reducible and if $F(D, D')$ is irreducible then $F(D, D')z = f(x, y)$ is called irreducible linear partial differential equation.

Working rule for finding *C.F.* of reducible Non Homogenous linear partial differential equation with constant coefficients : We have $F(D, D') = f(x, y)$

 Factorize $F(D, D')$ into linear factors

Type 1 : Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of *C.F.* is taken as

$$e^{\frac{cx}{b}} \phi(by + ax), \text{ if } b \neq 0 \qquad \qquad ......(1)$$

Now,

   (i) if $c = 0$, then (1) reduces to $\phi(by + ax)$, if $b \neq 0$

   (ii) if $a = 0$, then (1) reduces to $e^{\frac{cx}{b}} \phi(by)$, if $b \neq 0$

   (iii) if $a = c = 0$ and $b = 1$, then (1) reduces to $\phi(y)$

Type 2 : Corresponding to each repeated factor $(bD - aD' - c)^r$, the part of *C.F.* is taken as

$$e^{\frac{cx}{b}}\Big[\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax)\Big], \text{ if } b \neq 0 \qquad .....(2)$$

Now,

    (i)   If $c = 0$, then (2) reduces to

$$\phi_1(by+ax)+x\phi_2(by+ax)+x^2\phi_3(by+ax)+........+x^{r-1}\phi_r(by+ax),\ \text{if}\ \ b \neq 0$$

    (ii)   If $a = 0$, then (2) reduces to

$$e^{\frac{cx}{b}}\Big[\phi_1(by)+x\phi_2(by)+x^2\phi_3(by)+........+x^{r-1}\phi_r(by)\Big],\ \text{if}\ \ b \neq 0$$

    (iii)   If $a = c = 0$ and $b = 1$, then (2) reduces to

$$\phi_1(y)+x\phi_2(y)+x^2\phi_3(y)+........+x^{r-1}\phi_r(y)$$

Type 1 : Corresponding to each non-repeated factor $(bD-aD'-c)$, the part of *C.F.* is taken as

$$e^{-\frac{cy}{a}}\phi(by+ax)\ ,\ \text{if}\ a \neq 0 \qquad\qquad\qquad ......(3)$$

Now,

    (i) if $c = 0$, then (3) reduces to $\phi(by+ax)$, if $a \neq 0$

    (ii) if $b = 0$, then (3) reduces to $e^{-\frac{cy}{a}}\phi(ax)$, if $a \neq 0$

    (iii) if $b = c = 0$ and $a = 1$, then (3) reduces to $\phi(x)$

Type 2 : Corresponding to each repeated factor $(bD-aD'-c)^r$, the part of *C.F.* is taken as

$$e^{-\frac{cy}{a}}\Big[\phi_1(by+ax)+y\phi_2(by+ax)+y^2\phi_3(by+ax)+........+y^{r-1}\phi_r(by+ax)\Big],\ \text{if}\ \ a\neq 0 \qquad ...(4)$$

Now,

    (i)   If $c = 0$, then (4) reduces to

$$\phi_1(by+ax)+y\phi_2(by+ax)+y^2\phi_3(by+ax)+........+y^{r-1}\phi_r(by+ax),\ \text{if}\ \ a\neq 0$$

    (ii)   If $b = 0$, then (4) reduces to

$$e^{-\frac{cy}{a}}\Big[\phi_1(ax)+y\phi_2(ax)+y^2\phi_3(ax)+........+y^{r-1}\phi_r(ax)\Big],\ \text{if}\ \ a\neq 0$$

    (iii)   If $b = c = 0$ and $a = 1$, then (4) reduces to

$$\phi_1(x)+y\phi_2(x)+y^2\phi_3(x)+........+y^{r-1}\phi_r(x)$$

Method of finding *C.F.* of irreducible Linear Partial differential Equation with Constant Coefficients :

We have $F(D,D') = f(x,y)$ when the operator $F(D,D')$ is irreducible, it is not always possible to find a solution with as many arbitrary function as the order of $F(D,D')$, but we will develop a solution which contain as many arbitrary constants as we wish.

Method : Consider $F(D,D')z = 0$ ….(1)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

and $F(D, D')$ can not be factorize into linear factors solution (1) is $z = Ae^{hx+ky}$ where A, h, k are constants.

Now, $\qquad Dz = Ahe^{hx+ky}$ $\qquad\qquad\qquad\qquad D'z = Ake^{hx+ky}$

$\qquad\qquad D^2Z = Ae^{hx+ky}h^2$ $\qquad\qquad\qquad\qquad D'^2z = Ae^{hx+ky}k^2$

$\qquad\qquad$ .... $\qquad\qquad\qquad\qquad$ ....

$\qquad\qquad$ .... $\qquad\qquad\qquad\qquad$ ....

$\qquad\qquad D^rz = Ah^re^{hx+ky}$ $\quad$ ......(2) $\qquad D'^sz = Ak^se^{hx+ky}$ $\qquad\qquad\qquad$ ......(3)

and $\qquad D^rD'^sz = Ah^rk^se^{hx+ky}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ ......(4)

Substituting the values from (2), (3), (4) in (1), we have

$$Af(h,k)e^{hx+ky} = 0 \qquad\qquad\qquad ......(5)$$

Since $A \neq 0$, $e^{hx+ky} \neq 0$

if equation (5) in true then $f(h,k) = 0$

Now $\quad f(h,k) = 0$

If we take any value of h, we can able to find a value of k s.t. $f(h, k) = 0$, similarly if we take value of k, we can find value of h.

$\therefore$ We have infinite pair of h, k s.t. $f(h,k) = 0$, Thus $z = \sum_i A_i e^{h_ix + k_iy}$ is a solution of (1) and s.t. $f(h, k) = 0$

Example : Solve $(D^2 - D'^2 + D - D')z = 0$.

Solution : The given equations is

$$(D^2 - D'^2 + D - D')z = 0$$

$$\text{or} \qquad \left[(D^2 - D'^2) + (D - D')\right]z = 0$$

$$\text{or} \qquad \left[(D - D')(D + D') + (D - D')\right]z = 0$$

$$\text{or} \qquad (D - D')(D + D' + 1)z = 0$$

[In first factor $c = 0$, $b = 1$, $a = 1$

In 2nd factor $c = -1$, $b = 1$, $a = -1$]

Hence the required solution is

$$z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$$

$$\text{where } \phi_1 , \phi_2 \text{ are arbitrary functions.}$$

Exercise 8.1

Solve the following partial differential equations :

1. $(D - D' + 1)(D + 2D' - 3)z = 0$

2. $(D - 3D' - 2)^2 z = 0$

3. $(D^2 - a^2 D'^2 + 2abD + 2a^2 bD')z = 0$

4. $(D + 1)(D + D' - 1)z = 0$

5. $(s + p - q - z) = 0$

6. $(t + s + q) = 0$

7. $(D^2 - DD' + D' - 1)z = 0$

8. (a) $(D - D'^2)z = 0$ (b) $(D^2 - D')z = 0$

9. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} - n^2 z = 0$ [Hint. $h^2 + k^2 - n^2 = 0$, $h^2 + k^2 = n^2$ put $h = n\cos\alpha$, $h = n\sin\alpha$]

10. $(D^2 - D'^2 + D + 3D' - 2)z = 0$ Hint : $\begin{bmatrix} (D^2 - D'^2) + (2D + 2D') - D + D' - 2 = 0 \\ (D - D')(D + D') + 2(D + D') - D + D' - 2 = 0 \\ (D + D' - 1)(D - D' + 2) = 0 \end{bmatrix}$

11. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$ Hint : $\begin{bmatrix} D^2 - DD' - 2D'^2 + 2(D + D') \\ (D + D')(D - 2D') + 2(D + D') \\ (D + D')(D - 2D' + 2) \end{bmatrix}$

Answers

1. $z = e^{-x}\phi_1(y + x) + e^{3x}\phi_2(y - 2x)$

2. $z = e^{2x}\left[\phi_1(y + 3x) + x\phi_2(y + 3x)\right]$

3. $z = \phi_1(y - ax) + e^{-2abx}(y + ax)$

4. $z = e^{-x}\phi_1(y) + e^x \phi_2(y - x)$

5. $z = e^x \phi_1(y) + e^{-y}\phi_2(x)$

6. $z = \phi_1(-x) + e^{-x}\phi_2(y - x)$

7. $z = e^x \phi_1(y) + e^{-x}\phi_2(y + x)$

8. (a) $z = \sum a e^{k^2 x + ky}$

   (b) $\sum a e^{hx + h^2 y}$

9. $z = \sum a e^{hx \pm \sqrt{n^2 - h^2} y}$ or $\sum a e^{n(x\cos\alpha + y\sin\alpha)}$

10. $z = e^{-2x}\phi_1(y + x) + e^x \phi_2(y - x)$

11. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

8.2 General Solution of non homogenous linear Partial differential equation with constant co-efficients.

Let $\qquad\qquad\qquad\qquad F(D,D')z = f(x,y)$ $\qquad\qquad\qquad\qquad$ ......(1)

be a non homogenous linear partial differential equation with constant coefficient.

Let u be the complementary function of (1) i.e., u is solution of

$$F(D,D')z = 0$$

$\therefore \qquad\qquad\qquad\qquad F(D,D')u = 0$ $\qquad\qquad\qquad\qquad$ ......(2)

Now let v be the particular integral of (1)

$\therefore \qquad\qquad\qquad\qquad F(D,D')v = f(x,y)$ $\qquad\qquad\qquad\qquad$ ......(3)

Consider $\qquad F(D,D')(u+v) = F(D,D')u + F(D,D')v = 0 + f(x,y)$ $\qquad\qquad$ [Using (2) and (3)]

$\Rightarrow \qquad u + v$ is the solution of (1)

Hence the general solution of (1) is

$$z = C.F. + P.I.$$

Particular Integral of $\qquad F(D,D')z = f(x,y)$ is $\dfrac{1}{F(D,D')} f(x,y)$

We will follow the same rule for non Homogenous *P.D.E.* which we have discussed for Homogenous. P.D.E.

Case (1) When $\qquad\qquad f(x,y) = e^{ax+by}$ and $\quad F(a,b) \neq 0$

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$$

i.e., replace D to a and D' to b

Case (2) When $\qquad\qquad f(x,y) = \cos(ax+by)$ or $\quad \sin(ax+by)$

$\qquad\qquad$ or $\qquad\qquad \sin(ax+by)$

then $\qquad\qquad\qquad P.I. = \dfrac{1}{F(D,D')} \cos(ax+by)$ $\qquad$ OR $\qquad \dfrac{1}{F(D,D')} \sin(ax+by)$

obtained by putting $D^2 = -a^2$, $DD' = -ab$ and $D'^2 = -b^2$ provided denominator is non-zero.

Example : Solve $(D^2 - D'^2 + D - D')z = e^{2x+3y}$.

Solution : The given equation can be written as

$$[(D-D')(D+D')+(D-D')]z = e^{2x+3y}$$

or $$(D-D')(D+D'+1)z = e^{2x+3y}$$

Complementary function is

$$\phi_1(y+x)+e^{-x}\phi_2(y-x) \text{ , where } \phi_1 \text{ and } \phi_2 \text{ are arbitrary functions}$$

Now $P.I. = \dfrac{1}{(D-D')(D+D'+1)}e^{2x+3y} = \dfrac{1}{(2-3)(2+3+1)}e^{2x+3y} = -\dfrac{e^{2x+3y}}{6}$

Hence the complete solution is

$$z = \phi_1(y+x)+e^{-x}\phi_2(y-x)-\dfrac{e^{2x+3y}}{6}$$

Exercise 8.2

Solve the following partial differential equations :

1. $(D^2 - DD' - 2D)z = e^{2x+y}$

2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y}$

3. $(D^2 - D'^2 - 1)z = e^{x+y}$

4. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y}$

5. $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$

6. $(DD' + aD + bD' + ab)z = e^{mx+ny}$

7. $(D - D' - 1)(D - D' - 2)z = \sin(2x+3y)$

8. $(D^2 + D' + 4)z = e^{4x-y}$

9. $(D^2 + DD' + D' - 1)z = \sin(x+2y)$

10. $(D^2 - DD' + D' - 1)z = \cos(x+2y)$

Answers

1. $z = \phi_1(y)+e^{2x}\phi_2(y+x)-\dfrac{1}{2}e^{2x+y}$

2. $z = \phi_1(y-x)+e^{-2x}\phi_2(y+2x)-\dfrac{e^{2x+3y}}{10}$

3. $z = \sum ae^{hx+ky}-e^{x+y}$ where $h^2 - k^2 = 1$

4. $z = e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x)-\dfrac{e^{x-y}}{4}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 7

5. $z = e^x \phi_1(y+x) + e^{2x}\phi_2(y+x) + \dfrac{e^{2x-y}}{2}$

6. $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x) + \dfrac{1}{(m+b)(n+a)}e^{mx+ny}$

7. $z = e^x \phi_1(x+y) + e^{2x}\phi_2(y+x) + \dfrac{1}{10}\sin(2x+3y) - \dfrac{3}{10}\cos(2x+3y)$

8. $z = \sum a\, e^{hx-(h^2+4)y} + \dfrac{e^{4x-y}}{19}$ where a and h are arbitrary constant.

9. $z = e^{-x}\phi_1(y) + e^x \phi_2(y-x) - \dfrac{1}{10}[\cos(x+2y) + 2\sin(x+2y)]$

10. $z = e^x \phi_1(y) + e^{-x}\phi_2(y+x) + \dfrac{\sin(x+2y)}{2}$

8.3 Method to obtain Particular integral when $f(x,y) = x^m y^n$

$$P.I. = \frac{1}{F(D,D')} x^m y^n = [F(D,D')]^{-1} x^m y^n$$

We will expand $[F(D,D')]^{-1}$ in ascending power of $\dfrac{D'}{D}$ or $\dfrac{D}{D'}$. It should be noted that we shall get

different form of *P.I.* if we expand $[F(D,D')]^{-1}$ in ascending powers of $\dfrac{D'}{D}$ or $\dfrac{D}{D'}$,

It is better to expand $\left[F(D,D')\right]^{-1}$ in ascending powers of $\dfrac{D'}{D}$.

When $f(x,y) = Ve^{ax+by}$, when V is a function of x and y

In this case $P.I. = \dfrac{1}{F(D,D')} Ve^{ax+by} = e^{ax+by}\, \dfrac{1}{F(D+a,\ D'+b)} V$

Remark : when $f(x,y) = e^{ax+by}$, the $P.I. = \dfrac{1}{F(a,b)}\, e^{ax+by}$ provided $F(a,b) \neq 0$

But if $F(a, b) = 0$, then this result fail. But we shall tackle these problems with 2nd Method treating e^{ax+by} as $e^{ax+by}.1$ then

$$P.I.\ \frac{1}{F(D,D')} e^{ax+by}.1 = e^{ax+by}\, \frac{1}{f(D+a)(D'+b)}\ .1$$

which can evaluated by 1st Method treating 1 as $x^0 y^0$.

Exercise 8.3

Solve the following Partial differential equations .

1. $(D^2 - D')z = 2y - x^2$
2. $(2D^2 - D'^2 + D)z = x^2 - y$

3. $(D^2 - DD' + D)z = 1$
4. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy$

5. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$
6. $(D^2 - D'^2 + D + 3D' - 2)z = x^2 y$

7. $(D - 3D' - 2)^2 z = 2e^{2x} \sin(y + 3x)$
8. $(D - 1)(D - D' + 1)z = e^y$

9. $(D + D' - 1)(D + D' - 3)(D + D')z = e^{x+y} \sin(2x + y)$
10. $(D^2 - D')z = xe^{ax + a^2 y}$

11. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x + y)$

12. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2 y$

Answers

1. $z = \sum a e^{hx + h^2 y} + x^2 y$

2. $z = \sum a e^{hx + ky} - \dfrac{x^2 y^2}{2} + \dfrac{y^3}{6} - \dfrac{xy^4}{12} - \dfrac{y^4}{6} - \dfrac{y^6}{360}$ where h and k are connected by $2h^2 - k^2 + h = 0$

3. $z = \phi_1(y) + e^{-x}\phi_2(y + x) + x$

4. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) + \dfrac{x^2 y}{4} - \dfrac{xy}{4} - \dfrac{x^3}{12} + \dfrac{3x^2}{8} - \dfrac{x}{4}$

5. $z = e^x \phi_1(y - x) + e^{3x}\phi_2(y - 2x) + 6 + x + 2y$

6. $z = e^{-2x}\phi_1(y + x) + e^x \phi_2(y - x) - \dfrac{1}{8}[4x^2 y + 4xy + 6x^2 + 6y + 12x + 21]$

7. $z = e^{2x}[\phi_1(y + 3x) + x\phi_2(y + 3x)] + x^2 e^{2x} \sin(y + 3x)$

8. $z = e^x \phi_1(y) + e^{-x}\phi_2(y + x) - xe^y$

9. $z = e^x \phi_1(y - x) + e^{3x}\phi_2(y - x) + \phi_3(y - x) + \dfrac{e^{(x+y)}}{130}[3\cos(2x + y) - 2\sin(2x + y)]$

10. $z = \sum A e^{hx + h^2 y} + e^{ax + a^2 y}\left[\dfrac{x^2}{4a} - \dfrac{x}{4a^2}\right]$

11. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - \dfrac{1}{10}e^{2x+y} + \dfrac{x^2 y}{4} + \dfrac{3x^2}{8} - \dfrac{1}{4}xy - \dfrac{x}{2} - \dfrac{x^3}{12} - \dfrac{1}{6}\cos(2x + y)$

12. $z = e^{-2x}\phi_1(y + x) + e^x \phi_2(y - x) - \dfrac{e^{x-y}}{4} + \dfrac{x^2 y}{2} + \dfrac{xy}{2} + \dfrac{3}{4}x^2 + \dfrac{3}{4}y + \dfrac{3}{2}x + \dfrac{21}{8}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com Mob. 8607383607, 9802177766

Page 1

---------------- S C Q ----------------

1. The general solution of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is of the form

   1. $u = f(x+iy) + g(x-iy)$

   2. $u = f(x+y) + g(x-y)$

   3. $u = cf(x-iy)$

   4. $u = g(x+iy)$ **(GATE 1996)**

2. If $f(x)$ and $g(y)$ are arbitrary functions, then the general solution of the partial differential equation $u\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} = 0$ is given by

   1. $u(x,y) = f(x) + g(y)$

   2. $u(x,y) = f(x+y) + g(x-y)$

   3. $u(x,y) = f(x)g(y)$

   4. $u(x,y) = xg(y) + yf(x)$ **(GATE 2005)**

3. Let $u(x,t)$ be the bounded of

   $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$

   with $u(x,0) = \dfrac{e^{2x}-1}{e^{2x}+1}$. Then $\lim\limits_{t\to\infty} u(1,t)$ equals

   1. $-\dfrac{1}{2}$ 2. $\dfrac{1}{2}$

   3. -1 4. 1 **(GATE 2006)**

4. The general solution of the partial differential equation $\dfrac{\partial^2 z}{\partial x \partial y} = x + y$ is of the form

   1. $\dfrac{1}{2}xy(x+y) + F(x) + G(y)$

   2. $\dfrac{1}{2}xy(x-y) + F(x) + G(y)$

   3. $\dfrac{1}{2}xy(x-y) + F(x)G(y)$

   4. $\dfrac{1}{2}xy(x+y) + F(x)G(y)$ **(GATE 2010)**

5. A general solution of the second order equation $4u_{xx} - u_{yy} = 0$ is of the form $u(x,y) =$

   1. $f(x) + g(y)$

   2. $f(x+2y) + g(x-2y)$

   3. $f(x+4y) + g(x-4y)$

   4. $f(4x+y) + g(4x-y)$
   (CSIR NET SCQ June 2011)

6. The complete integral of the PDE

   $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = xe^{x+y}$ involving

   arbitrary function ϕ_1 and ϕ_2 is

   1. $\phi_1(y+x) + \phi_2(y+x) + \dfrac{1}{4}e^{x+y}$

   2. $\phi_1(y+x) + x\phi_2(y+x) + \dfrac{(x-1)}{4}e^{x+y}$

   3. $\phi_1(y-x) + \phi_2(y-x) + \dfrac{1}{4}e^{x+y}$

4. $\phi_1(y-x)+x\phi_2(y-x)+\dfrac{(x-1)}{4}e^{x+y}$

(CSIR NET SCQ Dec 2011)

7. The partial differential equation

$\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+u$ can be transformed to

$\dfrac{\partial v}{\partial t}=\dfrac{\partial^2 v}{\partial x^2}$. For

1. $v=e^{-t}u$ 2. $v=e^t u$

3. $v=tu$ 4. $v=-tu$

(CSIR NET SCQ Dec 2013)

8. Let $u(x,t)=e^{i\omega x}v(t)$ with $v(0)=1$ be a

solution to $\dfrac{\partial u}{\partial t}=\dfrac{\partial^3 u}{\partial x^3}$ then

1. $u(x,t)=e^{i\omega\left(x-\omega^2 t\right)}$ 2. $u(x,t)=e^{i\omega x-\omega^2 t}$

3. $u(x,t)=e^{i\omega\left(x+\omega^2 t\right)}$ 4. $u(x,t)=e^{i\omega^3(x-t)}$

(CSIR NET SCQ Dec 2014)

9. The PDE $\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}=x$, has

1. only one particular integral

2. a particular integral which is linear in x and y.

3. a particular integral which is a quadratic polynomial in x and y

4. more than one particular integral

(CSIR NET SCQ Dec 2015)

-------------------- M C Q --------------------

1. A bounded solution of the partial

differential $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+e^{-t}$ is

1. $u(x,t)=-e^{-t}$

2. $u(x,t)=e^{-x}e^{-t}$

3. $u(x,t)=e^{-x}+e^{-t}$

4. $u(x,t)=x-e^{-t}$

(CSIR NET MCQ Dec 2012)

2. If $u(x,t)$ satisfy the partial differential

equation $\dfrac{\partial^2 u}{\partial t^2}=4\dfrac{\partial^2 u}{\partial x^2}$ then $u(x,t)$ can be

of the form

1. $u(x,t)=f\left(e^{x-2t}\right)+g(x+2t)$

2. $u(x,t)=f\left(x^2-4t^2\right)+g\left(x^2+4t^2\right)$

3. $u(x,t)=f(2x-4t)+g(x+2t)$

4. $u(x,t)=f(2x-t)+g(2x+t)$

(CSIR NET MCQ Dec 2012)

3. If the initial value problem for partial

differential equation

$\dfrac{\partial u}{\partial t}+\dfrac{\partial^2 u}{\partial x^2}=0;\ u(x,0)=\sin(\pi x)$ has a

solution of the form $u(x,t)=\phi(t)\sin(\pi x)$,

then

1. ϕ is always negative

2. ϕ is always positive

3. ϕ is an increasing function

4. ϕ is a decreasing function

(CSIR NET MCQ Dec 2013)

4. Let $P(x,y)$ be a particular integral of the

partial differential equation

$\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial z}{\partial y}=2y-x^2$; then $P(2,3)$ equals

1. 2   2. 8
3. 12   4. 10

(CSIR NET MCQ Dec 2013)

5. Let (u,t) satisfy for $x\in\mathbb{R},\ t>0$,

$\dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial u}{\partial t}+2\dfrac{\partial^2 u}{\partial x^2}=0$. A solution of the

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

form $u = e^{ix} \, v(t)$ with $v(0) = 0$ and

$v'(0) = 1$

1. is necessarily bounded

2. satisfies $|u(x,t)| < e^t$

3. is necessarily unbounded

4. is oscillatory in x

(CSIR NET June 2014)

Answer Key

SCQ

1. 1	2. 3	3. 1
4. 1	5. 2	6. 4
7. 1	8. 1	9. 4

MCQ

1. 1	2. 1,3	3. 2,3
4. 3	5. 2,3,4	

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Chapter - 9

9.1 Classification of Linear Partial Differential Equations

Classification of linear Partial Differential Equation of second order in two independent variables : Consider a general partial differential equation of second order for a function of two independent variables x and y in the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \qquad ......(1)$$

where R, S, T are continuous function of x and y possessing partial derivatives defined in some domain D on the xy-plane.

equation (1) is said to be

1. **Hyperbolic** at a point (x, y) in Domain D, if $S^2 - 4RT > 0$

2. **Parabolic** at a point (x, y) in Domain D, if $S^2 - 4RT = 0$

3. **Elliptic** at a point (x, y) in Domain D, if $S^2 - 4RT < 0$

Remarks :

(i) We observe that the type of equation (1) is determined solely by its principal part i.e. ,
   $Rr + Ss + Tt$, which involve the highest order derivative of z.

(ii) If all R , S , T are constants , the differential equation will have the same nature throughout. If R , S , T are functions of x , y ; the same differential equation can be hyperbolic , parabolic or elliptic at different points of the region.

Note : Some author use u in place of z. In that case

$$r = \frac{\partial^2 u}{\partial x^2} , \qquad s = \frac{\partial^2 u}{\partial x \partial y}, \qquad t = \frac{\partial^2 u}{\partial y^2}$$

Example 1 : Classify one dimensional diffusion equation.

Solution : One dimensional diffusion equation is

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} \ \ i.e., \ \ r - q = 0$$

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = 1, \ \ S = 0, \ \ T = 0$

Now $\qquad \qquad \qquad \qquad S^2 - 4RT = 0 - 4.1.0 = 0$

$\therefore$ The given equation is parabolic.

Classification of a Partial Differential Equation in three independent variables : A linear partial differential equation of the second order in 3 independent variables x_1, x_2, x_3 is given by

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad \qquad ......(1)$$

where $a_{ij} \left(a_{ij} = a_{ji} \right)$, b_i, c are either constant or some function of the independent variables x_1, x_2, x_3 and u is dependent variable.

Since $a_{ij} = a_{ji}$, so the matrix $A = \left[a_{ij} \right]_{3\times3}$ given by equation (1) is a real symmetric matrix of order 3×3. The eigen values of matrix A are roots of the characteristic equation of A i.e., $|A - \lambda I| = 0$.

Now classification of (1) with the help of matrix A is

Types of Equation :

(i) If all the eigen values of A are non-zero and two of them have same sign , then equation (1) is known as hyperbolic type of equation.

(ii) If one of the eigen value is zero i.e. , $|A| = 0$, then equation (1) is known as parabolic type of equation .

(iii) If all the eigen values of A are non-zero and have same sign then (1) is known as elliptic type of equation.

Remark : We can remember the matrix A as

$$A = \begin{bmatrix} \text{coeff of } u_{xx} & \text{coeff of } u_{xy} & \text{coeff of } u_{xz} \\ \text{coeff of } u_{yx} & \text{coeff of } u_{yy} & \text{coeff of } u_{yz} \\ \text{coeff of } u_{zx} & \text{coeff of } u_{zy} & \text{coeff of } u_{zz} \end{bmatrix}$$

Here , A is a symmetric matrix.

Example 2 : Classify $u_{xx} + u_{yy} = u_{zz}$.

Solution : Matrix A of the given equation is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Eigen values of A are given by $|A - \lambda I| = 0$

$$\text{or} \qquad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\text{or} \qquad (1-\lambda)(1-\lambda)(1+\lambda) = 0, \qquad \lambda = 1, 1, -1$$

Here , all the eigen values are non-zero and two of them have same sign. Hence the given equation is of hyperbolic type.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

Classification of Partial differential Equation of second order in n-independent variables : A

linear Partial Differential equation with n variable $x_1, x_2, ..., x_n$ given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + c\,u = 0$$

where $a_{ij} = a_{ji}$, b_i, c are either constant or function of $x_1, x_2, ..., x_n$ and u is dependent variable.

Let $\qquad \delta_i = \dfrac{\partial}{\partial x_i}$, $\delta_i \, \delta_j = \dfrac{\partial^2}{\partial x_i \, \partial x_j} \qquad$ where $i = 1, 2, ... , n$ and $j = 1, 2, ..., n$

Now, consider the operator $\phi = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, \delta_i \, \delta_j$ for all non-zero real values of δ_i and δ_j positive or

negative at any point $(x_1, x_2, ..., x_n)$. Then , the differential equation (1) is said to be

(i) **Elliptic :** If ϕ is positive for all real values of δ_i and δ_j and it reduces to zero only when all

$\delta_i's$ and $\delta_j's$ are zero .

(ii) **Hyperbolic :** If ϕ can be both positive or negative.

(iii) **Parabolic :** If $\Delta = 0$, where $\qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{vmatrix}$

This method is more general than previous one , as it covered n independent variables

Exercise 9.1

Classify the given partial differential equations.

1. $\dfrac{\partial^2 z}{\partial x^2} + 4 \dfrac{\partial^2 z}{\partial x \, \partial y} + 4 \dfrac{\partial^2 z}{\partial y^2} = 0$
   $\qquad\qquad$ 2. $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$

3. $2\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + 5\dfrac{\partial^2 z}{\partial y^2} = 0$

4. Find where the partial differential equation $\dfrac{\partial^2 u}{\partial x^2} + t \dfrac{\partial^2 u}{\partial x \, \partial t} + x \dfrac{\partial^2 u}{\partial t^2} = 0$ is hyperbolic, parabolic and

   elliptic.

Answers

1. Parabolic $\qquad\qquad$ 2. Hyperbolic $\qquad\qquad$ 3. Elliptic

4. (i) Hyperbolic if $t^2 > 4x$ $\quad$ (ii) Parabolic if $t^2 = 4x$ $\quad$ (iii) Elliptic if $t^2 < 4x$

9.2 Reduction to canonical (normal) forms

Consider the linear partial differential equation of the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \qquad ......(1)$$

where R, S, T are continuous functions of x and y having continuous partial derivatives of as high an order as necessary.

Let the independent variables x and y be changed to u and v by means of transformations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \qquad \qquad ......(2)$$

so that the resulting equation in independent variables u and v, is transformed into one of the three canonical forms, which are easily integrable.

Now.
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

or
$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial x} \qquad ......(3)$$

$$\therefore \quad = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) \qquad \text{[using (3)]}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)\right]\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\frac{\partial z}{\partial v}\right]\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

<div align="right">[Using (3)]</div>

$$= \left[\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u^2} + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v\,\partial u}\right]\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \left[\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v^2}\right]\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$r = \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 z}{\partial u^2} + 2.\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

Similarly
$$t = \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

and
$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x\,\partial y}$$

$$= \left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\frac{\partial z}{\partial u}\right]\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x\,\partial y} + \left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\frac{\partial z}{\partial v}\right]\frac{\partial v}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x\,\partial y}$$

<div align="right">[using (3)]</div>

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

$$= \left[\frac{\partial u}{\partial x} \frac{\partial^2 z}{\partial u^2} + \frac{\partial v}{\partial x} \frac{\partial^2 z}{\partial u \, \partial v} \right] \frac{\partial u}{\partial y} + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial x \, \partial y} + \left[\frac{\partial u}{\partial x} \frac{\partial^2 z}{\partial u \, \partial v} + \frac{\partial v}{\partial x} \frac{\partial^2 z}{\partial v^2} \right] \frac{\partial v}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial x \, \partial y}$$

Thus $\quad s = \dfrac{\partial u}{\partial x} \dfrac{\partial u}{\partial y} \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial u \, \partial v} \left[\dfrac{\partial u}{\partial y} \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x} \dfrac{\partial v}{\partial y} \right] + \dfrac{\partial z}{\partial u} \dfrac{\partial^2 u}{\partial x \, \partial y} + \dfrac{\partial v}{\partial x} \dfrac{\partial v}{\partial y} \dfrac{\partial^2 z}{\partial v^2} + \dfrac{\partial z}{\partial v} \dfrac{\partial^2 v}{\partial x \, \partial y}$

Putting the value of p, q, r, s, t in equation (1) and simplifying, we get

$$A \frac{\partial^2 z}{\partial u^2} + 2B \frac{\partial^2 z}{\partial u \, \partial v} + C \frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right) = 0 \qquad \text{......(4)}$$

where $\qquad A = R\left(\dfrac{\partial u}{\partial x} \right)^2 + S \dfrac{\partial u}{\partial x} \dfrac{\partial u}{\partial y} + T\left(\dfrac{\partial u}{\partial y} \right)^2 \qquad \text{......(5)}$

$$B = R \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{1}{2} S \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + T \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \qquad \text{......(6)}$$

$$C = R\left(\frac{\partial v}{\partial x} \right)^2 + S \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y} \right)^2 \qquad \text{......(7)}$$

and $F\left(u, v, z, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v} \right)$ is transformed from of $f(x, y, z, p, q)$.

Now we shall determine u and v so that equation (4) reduces to simplest possible form. The method of evaluation of u and v become easy when the discriminant $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{......(8)}$$

is either positive, negative or zero everywhere.

The three cases are discussed separately as follows :

Case I : If $S^2 - 4RT > 0$ (i.e., the equation is hyperbolic).

In this case the root λ_1, λ_2 of equation (8) are real and distinct. We shall choose u and v such that the coefficients of $\dfrac{\partial^2 z}{\partial u^2}$ and $\dfrac{\partial^2 z}{\partial v^2}$ in the equation (4) vanish.

So let us take $\qquad\qquad\qquad \dfrac{\partial u}{\partial x} = \lambda_1 \dfrac{\partial u}{\partial y} \qquad\qquad \text{......(9)}$

and $\qquad\qquad\qquad\qquad \dfrac{\partial v}{\partial x} = \lambda_2 \dfrac{\partial v}{\partial y} \qquad\qquad \text{.....(10)}$

then
$$A = (R\lambda_1^2 + S\lambda_1 + T)\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

$R\lambda_1^2 + S\lambda_1 + T = 0$ as λ_1 is root of equation (8).

Similarly we can show $C = 0$

Now equation (9) can be written as

$$\frac{\partial u}{\partial x} - \lambda_1 \frac{\partial u}{\partial y} = 0 \text{ which is of the form } Pp + Qq = R$$

Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

By third ratio $\qquad du = 0$

$\Rightarrow \qquad u = c_1$, where c_1 is arbitrary constant.

Again taking first and second ratio , we get

$$-\lambda_1 \, dx = dy$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\lambda_1 \quad \text{or} \quad \frac{dy}{dx} + \lambda_1 = 0 \qquad \qquad .....(11)$$

Let $f_1(x, y) = c_2$ be the solution of equation (11)

$\therefore$ The solution of equation (9) is $u = f_1(x, y)$ $\qquad .....(12)$

which is suitable choice for u

Similarly the solution of equation (10) is

$$v = f_2(x, y) \qquad \qquad ......(13)$$

which is suitable choice for v.

Now it can be shown easily that

$$AC - B^2 = \frac{1}{4}\left[(4RT - S^2)\left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right)^2 \right] \qquad ......(14)$$

or $\qquad B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right)^2$ $\qquad$ [As $A = 0 = C$]

But in this case , $\qquad S^2 - 4RT > 0 \qquad \Rightarrow \qquad B^2 > 0$ $\qquad$ [From above equation]

Now equation (4) reduces to

$$2B\frac{\partial^2 z}{\partial u \, \partial v} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right) = 0 \qquad \qquad [\because \ A = C = 0]$$

As $B \neq 0$, dividing by $2B$, it reduces to $\frac{\partial^2 z}{\partial u \, \partial v} = \phi\left(u, v, z \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$ $\qquad ......(15)$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 7

which is the canonical form of given equation (1) and is simple to solve , than the given equation.

Case II : If $S^2 - 4RT = 0$ [i.e., the equation is parabolic]

In this case the roots of equation $R\lambda^2 + S\lambda + T = 0$ are real and equal i.e., $\lambda_1 = \lambda_2$

Here we choose u as in case I , such that

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \quad \text{which gives } u = f(x, y) \qquad \qquad .....(16)$$

and take v to be any function of x and y , which is independent of u.

∴   As in case I , $A = 0$

Also from (14) $AC - B^2 = 0$ $[\because S^2 - 4RT = 0]$

$\Rightarrow$ $B = 0$ $[\because A = 0]$

Here C can not be zero, otherwise v would be a function of u and consequently v would not be independent of u.

Putting $A = 0 = B$ in (4) and dividing by $C \neq 0$, it becomes

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right) \qquad \qquad .....(17)$$

Hence if we make the substitution $u = f(x, y)$ and v be any function of x and y , the given

equation (1) in this case reduces to the form $\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$ which is the canonical form

of equation (1) in this case and is simpler to solve , than the given equation.

Case III : If $S^2 - 4RT < 0$ [i.e., the given equation is elliptic]

In this case the root of the equation

$$R\lambda^2 + S\lambda + T = 0 \text{ are complex conjugates.}$$

Proceeding as in case I, here the equation (1) will reduces to the same canonical form.

i.e., $\frac{\partial^2 z}{\partial u \, \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$ as in case I but the variable u and v are not

real but the complex conjugate.

To obtain a real canonical form, we take

$$u = \alpha + i\beta, \quad v = \alpha - i\beta$$

Adding, $\alpha = \frac{1}{2}(u + v)$ and subtracting , $\beta = \frac{1}{2}i(v - u)$

Now we will further transform the independent variables u and v to α and β with these relations.

Now ,
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$\therefore$
$$\frac{\partial^2 z}{\partial u \, \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \, \partial \beta} - i\frac{\partial^2 z}{\partial \beta \, \partial \alpha} + \frac{\partial^2 z}{\partial \beta^2}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right)$$

Substituting in $\dfrac{\partial^2 z}{\partial u \, \partial v} = \phi\left(u, v, z, \dfrac{\partial z}{\partial u} \, , \, \dfrac{\partial z}{\partial v}\right)$

Canonical form of equation (1) in this case is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \, \beta, \, z, \, \frac{\partial z}{\partial \alpha} \, , \, \frac{\partial z}{\partial \beta}\right)$$

Type of the equation	Canonical form
When $\lambda_1 \neq \lambda_2$. Hyperbolic $S^2 - 4RT > 0$	$\dfrac{\partial^2 z}{\partial u \, \partial v} = \phi\left(u, v, z, \dfrac{\partial z}{\partial u} \, , \, \dfrac{\partial z}{\partial v}\right)$
$\lambda_1 = \lambda_2$ Parabolic $S^2 = 4RT$	$\dfrac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \dfrac{\partial z}{\partial u} \, , \, \dfrac{\partial z}{\partial v}\right)$
$\lambda_1 \neq \lambda_2$ Elliptic $S^2 - 4RT < 0$	$\dfrac{\partial^2 z}{\partial \alpha^2} + \dfrac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \, \beta, \, z\,\dfrac{\partial z}{\partial \alpha}, \, \dfrac{\partial z}{\partial \beta}\right)$ Here $u = \alpha + i\beta, \; v = \alpha - i\beta$

Working Rule for reducing a hyperbolic equation to the canonical form :

Steps :

1. Let the given hyperbolic equation be

$$Rr + Ss + Tt + f(x, \, y, \, z, \, p, \, q) = 0 \qquad\qquad ......(1)$$

then
$$S^2 - 4RT > 0$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com **Mob. 8607383607, 9802177766**

| Page 9 |

2. Write λ – quadratic $\quad R\lambda^2 + S\lambda + T = 0$ ......(2)

It will have two distinct roots , say λ_1 and λ_2 .

3. The corresponding characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

Solving these , we get $\qquad f_1(x, y) = c_1 \text{ and } f_2(x, y) = c_2$ ......(3)

4. Choose $\qquad u = f_1(x, y) \text{ and } v = f_2(x, y)$ ......(4)

Using relation (4) find p, q, r, s, t in terms of u and v as shown in last article

substituting the values of $p, q, r, s, t,$ obtained in step (4) in (1) and simplifying, we shall get the following canonical form.

$$\frac{\partial^2 z}{\partial u \ \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$$

Theorem : Consider the second order linear differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ......(1)$$

R,S and T are real constant and $\quad S^2 - 4RT > 0$ so that the equation is hyperbolic then there exist a transformation $u = u(x, y)$ and $v = v(x, y)$ of independent variable in (1) so that the transform equation in independent variable (u, v) may be written in the canonical form $\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$

Case I : If $R \neq 0$ such a transformation is given by $u = \lambda_1 x + y, v = \lambda_2 x + y,$ where λ_1 and λ_2 are the roots of the equation $R\lambda^2 + S\lambda + T = 0$

Case II : If $R = 0, S \neq 0, T \neq 0$ such a transformation is given by $u = x, v = x - \frac{S}{T} y$

Case III : If $R = 0, S \neq 0, T = 0$ such transformation is merely the identity transformation i.e. $u = x, v = y$

Example 1 : Reduce $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form .

Solution : Rewriting the given equation as

$$r - x^2 t = 0 \qquad ......(1)$$

and comparing (1) with $\qquad Rr + Ss + Tt + f(x, y, z, p, q) = 0$

we have $\qquad R = 1, S = 0, T = -x^2$

$\therefore$ $$S^2 - 4RT = 0 - 4(1)(-x^2) = 4x^2 > 0$$

Given equation is hyperbolic.

Now the quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$$\lambda^2 - x^2 = 0 \quad \Rightarrow \quad \lambda = \pm x$$

Let $\lambda_1 = x$ and $\lambda_2 = -x$ [Real and distinct]

$\therefore$ The corresponding characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

or $$\frac{dy}{dx} + x = 0 \quad \text{and} \quad \frac{dy}{dx} - x = 0$$

Integrating $$y + \frac{x^2}{2} = c_1 \quad \text{and} \quad y - \frac{x^2}{2} = c_2$$

Hence in order to reduce (1) in canonical, we change the independent variable x and y into u and v by

taking $$u = y + \frac{x^2}{2} \quad \text{and} \quad v = y - \frac{x^2}{2} \qquad \qquad ......(2)$$

Now $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v} \qquad \qquad ......(3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \qquad ......(4)$$

$\therefore$ $$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] \qquad \text{[using (3)]}$$

$$= x\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right] + 1.\left[\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= x\left[\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u\,\partial v}\right)x + \left(\frac{\partial^2 z}{\partial v\,\partial u} - \frac{\partial^2 z}{\partial v^2}\right)(-x)\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= x^2\left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \qquad \text{[using (4)]}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 11

$$= \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v} \right)1 + \left(\frac{\partial^2 z}{\partial v\,\partial u} + \frac{\partial^2 z}{\partial v^2} \right).1$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting the value of r and t in (1) i.e., $r - x^2 t = 0$, we have

$$x^2\left[\frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} \right] + \left[\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right] - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} \right) = 0$$

or
$$0 - 4x^2 \frac{\partial^2 z}{\partial u\,\partial v} = -\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

or
$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

or
$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \text{ as } u = y + \frac{x^2}{2}, \; v = y - \frac{x^2}{2}$$

$$\Rightarrow \qquad u - v = x^2$$

which is the required canonical form.

Working rule for reducing a parabolic equation to its Canonical form :

Steps :

1. Let the given parabolic equation be

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad\qquad ......(1)$$

then
$$S^2 - 4RT = 0$$

2. Write λ-quadratic $\qquad R\lambda^2 + S\lambda + T = 0 \qquad\qquad ......(2)$

It will have two equal roots.

3. Corresponding characteristic equation is $\lambda = \lambda_1$

$$\frac{dy}{dx} + \lambda_1 = 0$$

Solving it, we get $\qquad f_1(x, y) = c_1$, $\quad c_1$ is arbitrary constant . $\qquad ......(3)$

4. Choose $\qquad u = f_1(x, y) \quad$ and $\quad v = f_2(x, y) \qquad\qquad ......(4)$

where $f_2(x, y)$ is an arbitrary function of x and y and is independent of $f_1(x, y)$. For this verify

that Jacobian J of u and v of (4) is non-zero

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

5. Using relation (4) find p, q, r, s, t in terms of u and v. Substituting the values of p, q, r, s and t in (1) and simplifying , we get the canonical from of (1)

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u , v , z , \frac{\partial z}{\partial u} , \frac{\partial z}{\partial v} \right)$$

Theorem : Consider the second order linear differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \qquad ......(1)$$

R,S and T are real constant and $S^2 - 4RT = 0$ so that the equation is parabolic then there exist a

transformation $u = u(x, y)$ and $v = v(x, y)$ of the independent variable in (1) so that the transformed

equation in independent variable (u, v) may be written in the canonical form $\dfrac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v} \right)$

Case I : If $R \neq 0$ and $T \neq 0$ such a transformation is given by $u = \lambda x + y, v = y$, where λ is the

repeated real root of the quadratic equation $R\lambda^2 + S\lambda + T = 0$

Case II : If $R \neq 0$ and $T = 0$ such a transformation is given by $u = y, v = x$

Case III : If $R = 0, T = 0$ such transformation is merely the identity transformation i.e. $u = x, v = y$

Example 2 : Reduce $r - 6s + 9t + 2p + 3q - z = 0$ into canonical form .

Solution : Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

$$R = 1, \quad S = -6 \ T = 9 \quad f(x, y, z, p, q) = 2p + 3q - z$$

$$S^2 - 4RT = 36 - 4 \cdot 9 = 0$$

Here the given equation is parabolic.

The λ - quadratic is $\qquad \qquad R\lambda^2 + S\lambda + T = 0$

or $\qquad \qquad \lambda^2 - 6\lambda + 9 = 0$

or $\qquad \qquad (\lambda - 3)^2 = 0$

or $\qquad \qquad \lambda = 3 , 3$

The corresponding characteristic equation is

$$\frac{dy}{dx} + 3 = 0 \quad \text{or} \quad \frac{dy}{dx} = -3 \quad \text{or} \quad dy = -3dx$$

Integrating , we have $\qquad \qquad y + 3x = c_1 \qquad$ where c_1 is arbitrary constant.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 13

To reduce the given equation into canonical form , let us choose

$$u = y + 3x \text{ and } v = y$$

in such a manner that *u* and *v* are independent functions as verified below :

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 3 \neq 0$$

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = 3\dfrac{\partial z}{\partial u}$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(3\frac{\partial z}{\partial u}\right) = 3\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$= 3\left[\frac{\partial^2 z}{\partial u^2}\,3 + 0\right] = 9\frac{\partial^2 z}{\partial u^2}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$+ \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= 3\frac{\partial^2 z}{\partial u^2} + 0 + \frac{\partial^2 z}{\partial u\,\partial v}\,3 + 0 = 3\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v}\right]$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{dv}\right)$$

$$= \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= \left[\frac{\partial^2 z}{\partial u^2}\,1 + \frac{\partial^2 z}{\partial u\,\partial v}\right] + \left[\frac{\partial^2 z}{\partial u\,\partial v}\,1 + \frac{\partial^2 z}{\partial v^2}\,1\right]$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting the value of *p*, *q*, *r*, *s* , *t* in given equation , we have

$$9\frac{\partial^2 z}{\partial u^2} - 18\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v}\right) + 9\left[\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right] + 2\left(3\frac{\partial z}{\partial u}\right) + 3\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0$$

or $\qquad \dfrac{\partial^2 z}{\partial u^2}[9 + 9 - 18] + \dfrac{\partial^2 z}{\partial u\,\partial v}[-18 + 18] + \dfrac{\partial^2 z}{\partial v^2}[9] + \dfrac{\partial z}{\partial u}(6 + 3) + 3\dfrac{\partial z}{\partial v} - z = 0$

or $\qquad 9\dfrac{\partial^2 z}{\partial v^2} + 9\dfrac{\partial z}{\partial u} + 3\dfrac{\partial z}{\partial v} - z = 0$

or $\qquad \dfrac{\partial^2 z}{\partial v^2} = \dfrac{z}{9} - \dfrac{\partial z}{\partial u} - \dfrac{1}{3}\dfrac{\partial z}{\partial v}$ is required canonical form .

Working Rule for reducing an elliptic equation to its canonical form :

Steps :

1. Let the given elliptic equation be $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ ......(1)

   then $\qquad\qquad S^2 - 4RT < 0$

2. Write λ - quadratic $\quad R\lambda^2 + S\lambda + T = 0$ ......(2)

   It will have two roots , which are complex conjugates , say λ_1 and λ_2.

3. Corresponding characteristic equations are

   $$\frac{dy}{dx} + \lambda_1 = 0 \qquad \text{and} \qquad \frac{dy}{dx} + \lambda_2 = 0$$

   Solving these, we get

   $$f_1(x, y) + i\, f_2(x, y) = c_1 \qquad\qquad \text{and} \qquad f_1(x, y) - i\, f_2(x, y) = c_2 \quad ......(3)$$

4. Choose $\quad u = f_1(x, y) + i\, f_2(x, y) \qquad$ and $\quad v = f_1(x, y) - i\, f_2(x, y)$

   Let $\quad u = \alpha + i\beta$, $\quad v = \alpha - i\beta$

   so that $\qquad\qquad \alpha = f_1(x, y) \quad$ and $\quad \beta = f_2(x, y)$ ......(4)

5. Using relation (4) find p, q, r, s and t in terms of α and β.

6. Substituting the values of p, q, r, s, t and relation (4) in (1) and simplifying, we shall get the

   following canonical form $\dfrac{\partial^2 z}{\partial \alpha^2} + \dfrac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \dfrac{\partial z}{\partial \alpha}, \dfrac{\partial z}{\partial \beta}\right)$

Theorem : Consider the second order linear differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad\qquad ......(1)$$

R,S and T are real constant and $\quad S^2 - 4RT < 0$ so that the equation is elliptic then there exist a

transformation $u = u(x, y)$ and $v = v(x, y)$ of independent variable in (1) so that the transformed

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 15

equation in independent variable (u,v) may be written in the canonical form

$\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}\right)$ such a transformation is given by $u = ax + y$ and $v = bx$

where $a \pm ib\,(a$ and b real, $b \neq 0)$ are the conjugate complex roots of the quadratic

equation $R\lambda^2 + S\lambda + T = 0$.

Example 3 : Reduce $\dfrac{\partial^2 z}{\partial x^2} + x^2 \dfrac{\partial^2 z}{\partial y^2} = 0$ into canonical form.

Solution : Rewriting the given equation as $r + x^2 t = 0$ ......(1)

Comparing with $\qquad\qquad Rr + Ss + Tt + f(x, y, z, p, q) = 0$

we have $R = 1, \ S = 0, \ T = x^2$ so that $S^2 - 4RT = -4x^2 < 0$ showing that given equation is elliptic.

λ - quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + x^2 = 0$ giving $\lambda = \pm\, ix$

The corresponding characteristic equations are given by

$$\dfrac{dy}{dx} + ix = 0 \qquad \text{and} \qquad \dfrac{dy}{dx} - ix = 0$$

Integrating $\qquad y + i\left(\dfrac{x^2}{2}\right) = c_1 \qquad$ and $\qquad y - i\left(\dfrac{x^2}{2}\right) = c_2$

Choose $\qquad u = y + i\left(\dfrac{x^2}{2}\right) = \alpha + i\beta, \qquad$ and $\qquad v = y - i\dfrac{x^2}{2} = \alpha - i\beta$ ......(2)

where $\alpha = y, \ \beta = \dfrac{x^2}{2}$

Now u and v are new independent variables.

$\therefore \qquad p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial x} = 0 + x\dfrac{\partial z}{\partial \beta} = x\dfrac{\partial z}{\partial \beta}$ ......(3)

$q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial y} + \dfrac{dz}{\partial \beta}\dfrac{\partial \beta}{\partial y} = \dfrac{\partial z}{\partial \alpha} + 0 = \dfrac{\partial z}{\partial \alpha}$ or $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial \alpha}$ ......(4)

Now $\qquad r = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(x\dfrac{\partial z}{\partial \beta}\right) = \dfrac{\partial z}{\partial \beta}.1 + x.\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial \beta}\right)$

$\qquad\qquad = \dfrac{\partial z}{\partial \beta} + x\left[\dfrac{\partial}{\partial \alpha}\left(\dfrac{\partial z}{\partial \beta}\right)\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial}{\partial \beta}\left(\dfrac{\partial z}{\partial \beta}\right)\dfrac{\partial \beta}{\partial x}\right]$

$$= \frac{\partial z}{\partial \beta} + x \left[0 + \frac{\partial^2 z}{\partial \beta^2} \ x \right] = \frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2} \qquad \qquad \text{......(5)}$$

and $\qquad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \alpha} \right) = \frac{\partial^2 z}{\partial \alpha^2}$ $\qquad \qquad$ [using (4)]

Substituting the values of p, q, r, t in (1) , we have

$$\left[\frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2} \right] + x^2 \frac{\partial^2 z}{\partial \alpha^2} = 0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2} \frac{\partial z}{\partial \beta} = -\frac{1}{2\beta} \frac{\partial z}{\partial \beta} \qquad \left[\text{as } \beta = \frac{x^2}{2} \right]$$

which is the required canonical form of (1).

Exercise 9.2

Reduce the following equations to canonical forms :

1. $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$ $\qquad \qquad$ 2. $r + 2xs + x^2 t = 0$ $\qquad \qquad$ 3. $y^2 \dfrac{\partial^2 z}{\partial x^2} + x^2 \dfrac{\partial^2 z}{\partial y^2} = 0$

Answers

1. $2uv \dfrac{\partial^2 z}{\partial u \partial v} - v \dfrac{\partial z}{\partial v} = 0$ $\qquad \qquad$ 2. $\dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial z}{\partial u}$ $\qquad \qquad$ 3. $\dfrac{\partial^2 z}{\partial \alpha^2} + \dfrac{\partial^2 z}{\partial \beta^2} + \dfrac{1}{2} \left(\dfrac{1}{\alpha} \dfrac{\partial z}{\partial \alpha} + \dfrac{1}{\beta} \dfrac{\partial z}{\partial \beta} \right) = 0$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

Chapter - 10

10.1 Characteristic Equations for non-linear PDE

Characteristic equations and Characteristic curves :

Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \qquad ......(1)$$

Now , corresponding to this equation , consider the λ – quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad \qquad ......(2)$$

which has two roots. Then the ordinary differential equations $\dfrac{dy}{dx} + \lambda(x, y) = 0$ ......(3)

are called the characteristic equations.

The solutions of (3) are known as characteristic curves or simply the characteristics of the second

order partial differential equation (1).

Now consider the following 3 cases :

Case I : If $S^2 - 4RT > 0$, then (2) has two distinct real roots λ_1 , λ_2 (say), so we have two

characteristic equations $\dfrac{dy}{dx} + \lambda_1(x, y) = 0$ and $\dfrac{dy}{dx} + \lambda_2(x, y) = 0$

Solving these we get two distinct family of characteristics.

Case II : If $S^2 - 4RT = 0$, then equation (2) has two equal real roots , so we get only one characteristic

family of curve.

Case III : If $S^2 - 4RT < 0$, then equation (2) has complex roots. Hence there are no real

characteristics. Thus we get two families of complex characteristics.

Remark : There are two distinct , one or two complex characteristics according as the partial

differential equation is hyperbolic , parabolic or elliptic.

Example 1 : Find the characteristics of $y^2r - x^2t = 0$.

Solution : Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \text{ , we have } R = y^2, \, S = 0, \, T = -x^2$$

Now $\qquad S^2 - 4RT = 0 - 4(y^2)(-x^2) = 4x^2y^2 > 0$ $\qquad$ [for all non-zero x and y]

Hence (1) is hyperbolic everywhere except at $x = 0$ and $y = 0$

The λ – quadratic is $\qquad\qquad R\lambda^2 + S\lambda + T = 0$

$\qquad$ or $\qquad\qquad\qquad \lambda^2 y^2 - x^2 = 0$

$\qquad$ or $\qquad\qquad\qquad \lambda = \dfrac{x}{y}, -\dfrac{x}{y}$

$\therefore$ Corresponding characteristic equations are

$$\frac{dy}{dx} + \frac{x}{y} = 0 \qquad\qquad \text{and} \qquad\qquad \frac{dy}{dx} - \frac{x}{y} = 0$$

$\qquad$ or $\qquad x\,dx + y\,dy = 0 \qquad$ and $\qquad y\,dy - x\,dx = 0$

Integrating , $\qquad \dfrac{x^2}{2} + \dfrac{y^2}{2} = \dfrac{c_1}{2} \qquad$ and $\qquad \dfrac{y^2}{2} - \dfrac{x^2}{2} = \dfrac{c_2}{2}$

$\qquad$ or $\qquad x^2 + y^2 = c_1 \qquad$ or $\qquad (y^2 - x^2) = c_2$

which are required family of characteristics.

Example 2 : Find the real characteristics of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution : The given equation is $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

In symbolic form $\qquad\qquad x^2 r + 2xys + y^2 t = 0 \qquad\qquad$ ……(1)

Comparing given equation (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0,$ we get

$$R = x^2, S = 2xy, T = y^2$$

Now, $\qquad\qquad S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

Hence the equation (1) is parabolic every where.

Now, the quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$$x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$\Rightarrow \qquad\qquad (x\lambda + y)^2 = 0$

$\Rightarrow \qquad\qquad \lambda = -\dfrac{y}{x}, -\dfrac{y}{x}.$ Thus the roots are real and equal.

So, the corresponding characteristics equation is

$$\frac{dy}{dx} + \lambda = 0 \qquad \text{or} \qquad \frac{dy}{dx} - \frac{y}{x} = 0$$

$\qquad$ or $\qquad \dfrac{dy}{y} - \dfrac{dx}{x} = 0$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

Integrating, we get $\dfrac{y}{x} = c_1$ or $y = c_1 x$

which is the required family of characteristics. Thus, in this case we obtain one family of

characteristics representing a family of straight lines passing through the origin.

Example 3 : Find the real characteristics of $(1 + x^2)\dfrac{\partial^2 z}{\partial x^2} + (1 + y^2)\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

Solution : The given partial differential equation is

$$(1 + x^2)\dfrac{\partial^2 z}{\partial x^2} + (1 + y^2)\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$$

In symbolic form $(1 + x^2)r + (1 + y^2)t + xp + yq = 0$ ……(1)

Compare given equation (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$$R = 1 + x^2,\ S = 0,\ T = 1 + y^2$$

Now, $S^2 - 4RT = 0 - 4(1 + x^2)(1 + y^2) = -4(1 + x^2)(1 + y^2) < 0$

Hence, (1) is elliptic and it has no real characteristics.

Exercise 10.1

Find the real characteristics of the following partial differential equations :

1. $\dfrac{\partial^2 z}{\partial x^2} + 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = 0$ 2. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} - 8\dfrac{\partial^2 u}{\partial y^2} = 0$

3. $xy\dfrac{\partial^2 u}{\partial x^2} - (x^2 - y^2)\dfrac{\partial^2 u}{\partial x \partial y} - xy\dfrac{\partial^2 u}{\partial y^2} + y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} - 2(x^2 - y^2) = 0$.

4. For the partial differential equation $2\dfrac{\partial^2 u}{\partial x} - 2\dfrac{\partial^2 u}{\partial x \partial y} + 5\dfrac{\partial^2 u}{\partial y^2} = 0$ determine whether real

   characteristics exits or not.

Answers

1. $y - 2x = c$ 2. $y = 2x + c_1$, $y = -4x + c_2$ 3. $x^2 + y^2 = c_1$, $y = xc_2$

4. Characteristics are not real

J.R. INSTITUTE OF MATHEMATICS

---------------- **S C Q** ----------------

1. The second order PDE $u_{xx} - x^2 u_{yy} = 0$ then the canonical form of the PDE is

   1. $z_{uv} = \dfrac{1}{4(u+v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$

   2. $z_{uv} = \dfrac{1}{4(u+v)}\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

   3. $z_{uv} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$

   4. $z_{uv} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

2. The second order PDE

   $(n-1)^2 u_{xx} - y^{2n} u_{yy} = ny^{2n-1} u_y,$ where n is an integer then reduce canonical form is

   1. $z_{uu} = 0$ 2. $z_{vv} = 0$

   3. $z_{uv} = 0$ 4. $z_{uu} + z_{vv} = 0$

3. The second order partial differential equation $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$ is

   1. Hyperbolic   2. Elliptic

   3. Parabolic   4. None of these

4. The second order partial differential equation $yr + (x+y)s + xt = 0$ then reduce canonical form is

   1. $u\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial z}{\partial v} = 0$ as $u \neq 0$

   2. $u\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial z}{\partial u} = 0$ as $u \neq 0$

   3. $u\dfrac{\partial^2 z}{\partial u \partial v} + \left(\dfrac{\partial z}{\partial v}\right)^2 = 0$ as $u \neq 0$

   4. $u\dfrac{\partial^2 z}{\partial u \partial v} + \left(\dfrac{\partial z}{\partial u}\right)^2 = 0$ as $u \neq 0$

5. The second order partial differential equation $\dfrac{\partial^2 u}{\partial x^2} = (1+y)^2 \left(\dfrac{\partial^2 z}{\partial y^2}\right)$ then reduce canonical form is

   1. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial z}{\partial u}$

   2. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial z}{\partial v}$

   3. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{u}\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

   4. $4\dfrac{\partial^2 z}{\partial x \partial y} = \left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

6. The characteristic of the partial differential equation

   $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \cos^2 x \dfrac{\partial^2 z}{\partial y^2} + 2\dfrac{\partial z}{\partial x} + 3\dfrac{\partial z}{\partial y} = 0.$

   When it is of hyperbolic type …

   1. $\mathbb{R}$ 2. $\mathbb{R} - n\pi$

   3. $\mathbb{R} - (2n+1)\dfrac{\pi}{2}$ 4. None

   (GATE 1997)

7. The equation $x^2(y-1)Z_{xx} - x(y^2-1)Z_{xy} + y(y^2-1)Z_{yy} + Z_x = 0$ is hyperbolic in the entire xy-plane except along

   1. x-axis   2. y-axis

   3. A line parallel to y-axis

   4. A line parallel to x-axis

   (GATE 2000)

8. The characteristics curves of the equation

$x^2 u_{xx} - y^2 u_{yy} = x^2 y + x;\ x > 0,\ u = (x, y)$ are

1. rectangular hyperbola
2. parabola
3. circle
4. straight line   **(GATE 2000)**

9. Pick the region in which the following

differential equation is hyperbolic

$y u_{xx} + 2xy u_{xy} + x u_{yy} = u_x + u_y$

1. $xy \neq 1$ 2. $xy \neq 0$

3. $xy > 1$ 4. $xy > 0$

(GATE 2003)

10. The partial differential equation

$$x \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial x} = 0$$

is

1. Elliptic in the region $x < 0,\ y < 0,\ xy > 1$

2. Elliptic in the region $x > 0,\ y > 0,\ xy > 1$

3. Parabolic in the region

$x < 0,\ y < 0,\ xy > 1$

4. Hyperbolic in the region

$x < 0,\ y < 0,\ xy > 1$ **(GATE 2005)**

11. In the region $x > 0,\ y > 0$, the partial

differential equation

$$\left(x^2 - y^2\right)\frac{\partial^2 u}{\partial x^2} + 2\left(x^2 + y^2\right)\frac{\partial^2 u}{\partial x \partial y}$$

$$+\left(x^2 - y^2\right)\frac{\partial^2 u}{\partial y^2} = 0$$

1. Changes type   2. is elliptic
3. is parabolic   4. is hyperbolic

(GATE 2006)

12. If the partial differential equation

$(x-1)^2 u_{xx} - (y-2)^2 u_{yy} + 2x u_x + 2y u_y$

$+2xyu = 0$ is parabolic in $S \subseteq R^2$ but not

in $R^2 \setminus S$, then S is

1. $\left\{(x, y) \in R^2 : x = 1 \text{ or } y = 2\right\}$

2. $\left\{(x, y) \in R^2 : x = 1 \text{ and } y = 2\right\}$

3. $\left\{(x, y) \in R^2 : x = 1\right\}$

4. $\left\{(x, y) \in R^2 : y = 2\right\}$ **(GATE 2008)**

13. The partial differential equation

$$x^2 \frac{\partial^2 z}{\partial x^2} - \left(y^2 - 1\right)x \frac{\partial^2 z}{\partial x \partial y} + y\left(y - 1\right)^2 \frac{\partial^2 z}{\partial y^2}$$

$$+x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \text{ is hyperbolic in a region}$$

in the XY-plane if

1. $x \neq 0$ and $y = 1$ 2. $x = 0$ and $y \neq 1$

3. $x \neq 0$ and $y \neq 1$ 4. $x = 0$ and $y = 1$

(GATE 2011)

14. The partial differential equation

$$x \frac{\partial^2 u}{\partial x^2} + (x - y)\frac{\partial^2 u}{\partial x \partial y} - y \frac{\partial^2 u}{\partial y^2} + \frac{1}{4}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right) = 0$$

is

1. Hyperbolic along the line $x + y = 0$

2. Elliptic along the line $x - y = 0$

3. Elliptic along the line $x + y = 0$

4. Parabolic along the line $x + y = 0$

(GATE 2017)

15. The number of characteristic curves of the

PDE $\left(x^2 + 2y\right)u_{xx} + \left(y^3 - y + x\right)u_{yy}$

$+x^2\left(y - 1\right)u_{xy} + 3u_x + u = 0$ passing

through the point $x = 1,\ y = 1$ is

1. 0   2. 1   3. 2   4. 3

(CSIR NET SCQ June 2011)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

16. The second order partial differential

equation

$$\frac{(x-y)^2}{4}\frac{\partial^2 u}{\partial x^2} + (x-y)\sin(x^2+y^2)\frac{\partial^2 u}{\partial x\partial y}$$

$$+\cos^2(x^2+y^2)\frac{\partial^2 u}{\partial y^2} + (x-y)\frac{\partial u}{\partial x}$$

$$+\sin^2(x^2+y^2)\frac{\partial u}{\partial y} + u = 0 \text{ is}$$

1. Elliptic in the region

$$\left\{(x,y): x \neq y,\ x^2+y^2 < \frac{\pi}{6}\right\}$$

2. Hyperbolic in the region

$$\left\{(x,y): x \neq y,\ \frac{\pi}{4} < x^2+y^2 < \frac{3\pi}{4}\right\}$$

3. Elliptic in the region

$$\left\{(x,y): x \neq y,\ \frac{\pi}{4} < x^2+y^2 < \frac{3\pi}{4}\right\}$$

4. Hyperbolic in the region

$$\left\{(x,y): x \neq y,\ x^2+y^2 < \frac{\pi}{4}\right\}$$

(CSIR NET SCQ Dec 2011)

17. The second order PDE $u_{yy} - yu_{xx} + x^3 u = 0$

is

1. elliptic for all $x \in \mathbb{R},\ y \in \mathbb{R}$

2. parabolic for all $x \in \mathbb{R},\ y \in \mathbb{R}$

3. elliptic for all $x \in \mathbb{R},\ y < 0$

4. hyperbolic for all $x \in \mathbb{R},\ y < 0$

(CSIR NET SCQ June 2012)

18. The partial differential equation

$$y\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} = 0 \text{ is hyperbolic in}$$

1. the second and fourth quadrants

2. the first and second quadrants

3. the second and third quadrants

4. the first and third quadrants

(CSIR NET SCQ Dec 2012)

19. The partial differential equation

$$\frac{\partial^2 u}{\partial y^2} - y\frac{\partial^2 u}{\partial x^2} = 0 \text{ has}$$

1. two families of real characteristic

curves for $y < 0$

2. no real characteristics for $y > 0$

3. vertical lines as a family of

characteristic curves for $y = 0$

4. branches of quadratic curves as

characteristics for $y \neq 0$

(CSIR NET SCQ June 2013)

20. Let a, b, c be continuous functions defined

on $\mathbb{R}^2$. Let v_1, v_2, v_3 be nonempty subset of

$\mathbb{R}^2$ such that $v_1 \cup v_2 \cup v_3 = \mathbb{R}^2$ and the

PDE $a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = 0$

is elliptic in v_1, parabolic in v_2 and

hyperbolic v_3, then

1. v_1, v_2 and v_3 are open sets in $\mathbb{R}^2$

2. v_1 and v_3 are open sets in $\mathbb{R}^2$

3. v_1 and v_2 are open sets in $\mathbb{R}^2$

4. v_2 and v_3 are open sets in $\mathbb{R}^2$

(CSIR NET SCQ Dec 2013)

21. Let a, b, c, d be four differential functions

defined on $\mathbb{R}^2$. Then the partial differential

equation $\left(a\left(x,y\right)\dfrac{\partial}{\partial x}+b\left(x,y\right)\dfrac{\partial}{\partial y}\right)$

$\left(c\left(x,y\right)\dfrac{\partial}{\partial x}+d\left(x,y\right)\dfrac{\partial}{\partial y}\right)u=0$ is

1. always hyperbolic

2. always parabolic

3. never parabolic

4. never elliptic

(CSIR NET SCQ June 2016)

22. The PDE $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial y^2}=0$ is

1. hyperbolic for $x>0,\ y<0$

2. elliptic for $x>0,\ y<0$

3. hyperbolic for $x>0,\ y>0$

4. elliptic for $x<0,\ y>0$

(CSIR NET SCQ Dec 2016)

23. Let D denote the unit disc given by $\left\{(x,y)\mid x^2+y^2\le1\right\}$ and let D^c be its complement in the plane. The partial differential equation

$\left(x^2-1\right)\dfrac{\partial^2 u}{\partial x^2}+2y\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\partial^2 u}{\partial y^2}=0$ is

1. parabolic for all $(x,y)\in D^c$

2. hyperbolic for all $(x,y)\in D$

3. hyperbolic for all $(x,y)\in D^c$

4. parabolic for all $(x,y)\in D$

(CSIR NET Dec 2017)

-------------------- M C Q -------------------- -

1. The PDE is $\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}=0$ is

1. Parabolic and has characateristics $\xi\left(x,y\right)=x+2y,\ \eta\left(x,y\right)=x-2y$

2. Reducible to the canonical form

$\dfrac{\partial^2 u}{\partial\xi^2}=0$, where $\xi\left(x,y\right)=x+2y$

3. Reducible to the canonical form

$\dfrac{\partial^2 u}{\partial\eta^2}=0$, where $\eta\left(x,y\right)=x+2y$

4. Parabolic and has the general solution $u=\left(x-y\right)f_1\left(x+y\right)+f_2\left(x-y\right)$, where f_1,f_2 are arbitrary functions.

(CSIR NET MCQ June 2014)

2. The second order partial differential

equation $u_{xx}+xu_{yy}=0$ is

1. elliptic for $x>0$

2. hyperbolic for $x>0$

3. elliptic for $x<0$

4. hyperbolic for $x<0$

(CSIR NET June 2015)

3. A solution of the PDE

$x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2-u=0$

represents

1. an ellipse in the x-y plane

2. and ellipsoid in the xyu space

3. a parabola in the u-x plane

4. a hyperbola in the u-y plane

(CSIR NET MCQ Dec 2015)

4. Consider the second order PDE

$8\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial^2 z}{\partial x\partial y}-3\dfrac{\partial^2 z}{\partial y^2}=0$

Then which of the following are correct ?

1. the equation is elliptic

2. the equation is hyperbolic

3. the general solution is

$z=f\left(y-\dfrac{x}{2}\right)+g\left(y+\dfrac{3x}{4}\right)$, for

arbitrary differentiable functions f and g

4. the general solution is

$z=f\left(y+\dfrac{x}{2}\right)+g\left(y-\dfrac{3x}{4}\right)$, for

arbitrary differentiable functions f and g

(CSIR NET Dec 2017)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 5

Answer Key

SCQ

1. 3	2. 3	3. 2
4. 1	5. 4	6. 2
7. 2	8. 1	9. 3
10. 4	11. 4	12. 2
13. 3	14. 4	15. 1
16. 2	17. 3	18. 1
19. 3	20. 2	21. 4
22. 1	23. 3	

MCQ

1. 3	2. 1,4	3. 3
4. 2,3		

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

Chapter - 11

11.1 Heat Wave and Laplace equations

Introduction : In physical problem we always seek a solution of the differential equations which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a boundary value problem.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary value problems involving linear partial differential equations can be solved by the following method.

A solution which breaks up into a product of functions each of which contains only one of the variables. The following explains this method.

Method of separation of variables :

Example 1 : Using the method of separation of variables, solve $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$

Solution : Assume the solution $u(x,t) = X(x)T(t)$ substituting in the given equation, we have

$$X'T = 2XT' + XT \qquad \Rightarrow \qquad (X'-X)T = 2XT' \qquad \Rightarrow \qquad \frac{X'-X}{2X} = \frac{T'}{T} = k \ \text{(say)}$$

$$\therefore \ X'-X-2kX = 0 \qquad \Rightarrow \qquad \frac{X'}{X} = 1+2k \qquad \dots(1) \ \text{and} \qquad \frac{T'}{T} = k \qquad \dots(2)$$

Solving (1), $\log X = (1+2k)x + \log c \qquad \Rightarrow \qquad X = c\,e^{(1+2k)x}$

From (2), $\log T = kt + \log c_1 \qquad \Rightarrow \qquad T = c_1 e^{kt}$

Thus $u(x,t) = XT = cc_1 e^{(1+2k)x} \cdot e^{kt}$ ....(3)

Now $6e^{-3x} = u(x,0) = cc_1 e^{(1+2k)x}$

$\Rightarrow \quad cc_1 = 6$ and $1+2k = -3$ or $k = -2$ substituting these values in (3), we get $u(x,t) = 6e^{-3x}e^{-2t}$

$\Rightarrow \quad u(x,t) = 6e^{-(3x+2t)}$ which is the required solution.

Some Important Equations :

1. Heat (Diffusion) Equation : $\nabla^2 u = \left(\dfrac{1}{k}\right)\left(\dfrac{\partial u}{\partial t}\right)$

One dimensional : $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}$

Two dimensional : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}$

Three dimensional : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}$

Note : Sometimes we write c^2 instead of K in heat equation.

Remark : ∇^2 is called the Laplacian operator and is defined as $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$. A function u

which satisfies Laplace's equation, is called the harmonic function.

2. Wave Equation : $\nabla^2 u = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$

One dimensional : $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$

Two dimensional : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$

Three dimensional : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$

3. Laplace's (or harmonic) Equation : $\nabla^2 u = 0$

Two dimensional : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Three dimensional : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

Solution of Heat Equation : Given $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}$ ……(1)

Let solution of (1) is of the form $u(x,t) = X(x)T(t)$ …….(2)

Where X is a function of x alone and T is a function of t alone.

Using (2) in (1), we have $X''T = \dfrac{1}{k}XT' \ \Rightarrow \ \dfrac{X''}{X} = \dfrac{1}{k}\dfrac{T'}{T}$ …….(3)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 3

Clearly the L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. Since x and t are independent variables, (3) can hold good if each side is equal to a constant, say u. Then (3) leads to

$$X'' - uX = 0 \qquad\qquad\qquad\qquad ..\dots..(4)$$

$$\text{and} \quad T' = ukT \qquad\qquad\qquad\qquad \dots\dots(5)$$

Three cases arises.

Case I : Let $u = 0$. Then solutions of (4) and (5) are $X = a_1 x + a_2$ and $T = a_3$ $\qquad \dots\dots(6)$

Case II : Let u be positive, say λ^2, where $\lambda \neq 0$. Then (4) and (5) becomes $X'' - \lambda^2 X = 0$ and

$$T' = \lambda^2 kT \qquad \Rightarrow \qquad X = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \text{ and } T = b_3 e^{\lambda^2 kt} \qquad\qquad \dots\dots(7)$$

Case III : Let u be negative, say $-\lambda^2$, $\lambda \neq 0$. Then (4) and (5) becomes $X'' + \lambda^2 X = 0$ and

$$T' = -\lambda^2 kT \qquad\qquad \Rightarrow \qquad X = c_1 \cos \lambda x + c_2 \sin \lambda x \text{ and } T = c_3 e^{-\lambda^2 kt} \qquad \dots\dots(8)$$

Thus the various possible solutions are

$$u(x,t) = A_1 x + A_2 \qquad\qquad\qquad\qquad \dots\dots(9)$$

$$u(x,t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{-\lambda^2 kt} \qquad\qquad\qquad \dots\dots(10)$$

$$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\lambda^2 kt} \qquad\qquad\qquad \dots\dots(11)$$

where $A_1 = a_1 a_3$, $A_2 = a_2 a_3$, $B_1 = b_1 b_3$, $B_2 = b_2 b_3$, $c_1 = c_1 c_3$, $c_2 = c_2 c_3$ are new arbitrary constants.

Now we have to choose that solution which is consistent with the physical nature of problems. Since we are dealing with problem of heat conduction, temperature $u(x,t)$ must decrease with the increase of time. Accordingly the solution given by (11) is the only suitable solution.

Working Rule for solving heat equation when both the ends of a bar of length a are kept at temperature zero and the initial temperature $f(x)$ is prescribed :

Step I : Solution of heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K} \dfrac{\partial u}{\partial t}$ $\qquad\qquad\qquad \dots\dots(1)$

Subject to the boundary conditions $u(0,t) = u(a,t) = 0$, for all t $\qquad\qquad \dots\dots(2)$

and the initial condition $u(x,0) = f(x)$, $\quad 0 < x < a$ $\qquad\qquad \dots\dots(3)$

is given by $u(x,t) = \displaystyle\sum_{n=1}^{\infty} E_n \sin\left(\dfrac{n\pi x}{a}\right) e^{-c_n^2 t}$ $\qquad\qquad\qquad \dots\dots(4)$

where $E_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\sin\left(\dfrac{n\pi x}{a}\right)dx,\ n=1,2,3,......$ ......(5)

and $c_n^2 = \dfrac{n^2\pi^2 k}{a^2}$ ......(6)

Step II : Compare the given problem with (1), (2) and (3) and find particular values of k, a and $f(x)$.

Step III : Substitute the particular values of k, a and $f(x)$ in (5) and (6) to get E_n and a_n^2 at the desired solution of the given boundary value problem.

Working rule for solving heat equation when both the ends of a bar of length a are insulated and the initial temperature $f(x)$ is prescribed :

Step I : The solution of the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}$ .....(1)

Subject to the boundary conditions $u_x(0,t) = u_x(a,t) = 0$, for all t .....(2)

And the initial condition $u(x,0) = f(x)$, for all x .....(3)

is given by $u(x,t) = \dfrac{E_0}{2} + \displaystyle\sum_{n=1}^{\infty} E_n\cos\dfrac{n\pi x}{a}e^{-c_n^2 t}$ .....(4)

where $E_0 = \dfrac{2}{a}\displaystyle\int_0^a f(x)\,dx$, $\qquad E_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\cos\dfrac{n\pi x}{a}dx,\quad n=1,2,3,......$ .....(5)

and $\qquad c_n^2 = \dfrac{n^2\pi^2 k}{a^2}$ .....(6)

Step II : Compare the given problem with (1), (2) and (3) and find particular values of k, a and $f(x)$.

Step III : Substitute the particular values of k, a and $f(x)$ in (5) and (6) and calculate E_0, E_n and c_n^2.

Step IV : Substitute the values of coefficients E_0, E_n and c_n^2 obtained in step III in (4) to arrive at the desired solution of the given boundary value problem.

Example 2: Solve the one-dimensional diffusion $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial u}{\partial t},\quad 0\le x\le 2\pi,\quad t\ge 0$ .....(1)

Subject to the boundary conditions : $u(x,0) = \sin^3 x$ for $0\le x\le 2\pi$ .....(2)

and $u(0,t) = u(2\pi,t) = 0$ for $t\ge 0$ .....(3)

Solution : Solution of one dimensional wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} E_n\sin\left(\dfrac{n\pi x}{a}\right)e^{-c_n^2 t},\quad n=1,2,3,...$$ .....(4)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 5

where $c_n^2 = \dfrac{n^2 \pi^2 k}{a^2}$ ......(5)

$$E_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx, \quad n = 1, 2, 3, \ldots \quad \text{......(6)}$$

Comparing (4), (5) , (6) with (1), (2), (3), we get $a = 2\pi, \ f(x) = \sin^3 x, \ c_n^2 = \dfrac{n^2 k}{4}$

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{nx}{2}\right) e^{-n^2 k \frac{t}{4}}, \quad n = 1, 2, 3, \ldots$$

Now $u(x,0) = \sin^3 x \qquad \Rightarrow \qquad \displaystyle\sum_{n=1}^{\infty} E_n \sin\left(\frac{nx}{2}\right) = \sin^3 x = \frac{1}{4}[3 \sin x - \sin 3x]$

$$\Rightarrow \qquad E_1 \sin \frac{x}{2} + E_2 \sin x + E_3 \sin \frac{3x}{2} + E_4 \sin 2x + E_5 \sin \frac{5x}{2} + E_6 \sin 3x + \ldots = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \qquad \ldots(7)$$

Equating the coefficients of like terms on both sides of (7) we get

$E_2 = \dfrac{3}{4}, \ E_6 = -\dfrac{1}{4}$ and $E_n = 0$ when $n \neq 2$ or $n \neq 6$ substituting these values in , we get

$$u(x,t) = E_2 \sin x e^{-kt} + E_6 \sin 3x e^{-9kt} = \frac{3}{4} \sin x e^{kt} - \frac{1}{4} \sin 3x e^{-9kt}$$

Example 3 : Find the solution of one-dimensional diffusion equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K} \dfrac{\partial u}{\partial t}$ satisfying the

following boundary conditions.

(i) u is bounded as $t \to \infty$

(ii) $u_x(0,t) = 0, \ u_x(a,t) = 0$ for all t

(iii) $u(x,0) = x(a-x), \ \ 0 < x < a$

Solution : We know that the bounded solution of the diffusion equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{K} \dfrac{\partial u}{\partial t}$ .....(1)

Subject to boundary condition $u_x(0,t) = u_x(a,t) = 0$ for all t ....(2)

and the initial condition $u(x,0) = f(x), \ \ 0 < x < a$ ....(3)

is given by $\qquad u(x,t) = \dfrac{E_0}{2} + \displaystyle\sum_{n=1}^{\infty} \cos \frac{n\pi x}{a} e^{-c_n^2 t}$ ....(4)

where $E_0 = \dfrac{2}{a}\displaystyle\int_0^a f(x)\,dx,$

$$E_n = \dfrac{2}{a}\int_0^a f(x)\cos\dfrac{n\pi x}{a}\,dx, \ \ n = 1,2,3,....\Bigg\} \qquad\qquad ……(5)$$

and $\qquad c_n^2 = \dfrac{n^2\pi^2 K}{a^2}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad ……(6)$

comparing the given BVP with the BVP given by (1), (2) and (3) we have $k = k, \ a = a,$ and

$f(x) = ax - x^2$ so from (5) we have

$$E_0 = \frac{2}{a}\int_0^a (ax - x^2)\,dx = \frac{2}{a}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{2}{a}\left[\frac{a^3}{2} - \frac{a^3}{3}\right] = \frac{a^2}{3}$$

$$E_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\frac{n\pi x}{a}\,dx = \frac{2}{a}\left[(ax - x^2)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (a - 2x)\left(\frac{-\cos(n\pi x/a)}{n^2\pi^2/a^2}\right) + (-2)\left(\frac{-\sin(n\pi x/a)}{n^3\pi^3/a^3}\right)\right]_0^a$$

$$= \frac{2}{a}\left[-a\times\frac{a^2}{n^2\pi^2}(-1)^n - a\times\frac{a^2}{n^2\pi^2}\right] = -\frac{2a^2}{n^2\pi^2}\left\{1 + (-1)^n\right\}$$

Hence if $n = 2m$, then $E_n = E_{2m} = -(a^2/m^2\pi^2)$ and if $n = 2m - 1$, then $E_n = E_{2m-1} = 0$, also

$c_n^2 = \dfrac{n^2\pi^2 k}{a^2} = \dfrac{4m^2\pi^2 k}{a^2}$, if $n = 2m$ substituting the above values of E_0, E_n and c_n^2 in (4), the required

solution is given by $u(x,t) = \dfrac{a^2}{6} + \displaystyle\sum_{m=1}^{\infty}\left(-\dfrac{a^2}{m^2\pi^2}\right)\cos\dfrac{2m\pi x}{a}\,e^{-\left(4m^2\pi^2 kt\right)/a^2}$

Working rule for solving Non-Homogeneous heat equation :

$u_t - k u_{xx} = f(x,t) \qquad 0 < x < a : t > 0$

$u(0,t) = u(a,t) = 0 \qquad\qquad\qquad u(x,0) = \phi(x)$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi}{a}\,x\,e^{-k\left(\frac{n^2\pi^2}{a^2}\right)t} + \sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\int_0^a e^{-k\frac{n^2\pi^2}{a^2}(s-t)}f_n(s)\,ds}$$

$$\boxed{E_n = \frac{2}{a}\int_0^a \phi(x)\sin\frac{n\pi x}{a}\,dx, \ f_n(t) = \frac{2}{a}\int_0^a f(x,t)\sin\frac{n\pi x}{a}\,dx}$$

Example : $u_t - u_{xx} = e^{-t}\sin 3x \qquad 0 < x < \pi, \ t > 0$

$u(0,t) = u(\pi,t) = 0 \qquad u(x,0) = \sin x \qquad 0 < x < \pi$

Solution : $E_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\sin x \sin nx\,dx \qquad\qquad E_1 = \dfrac{\pi}{2}\times\dfrac{2}{\pi} = 1 \qquad\qquad E_n = 0 \qquad n = 2,3$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 7

$$f_n(t) = \frac{2}{\pi}\int_0^\pi e^{-t}\sin 3x \sin nx\, dx = \frac{2}{\pi}\left[e^{-t}\int_0^\pi \sin 3x \sin n\,x\, dx\right]$$

$$f_3(t) = \frac{2}{\pi}\left[e^{-t}\frac{\pi}{2}\right]$$

$$f_3(t) = e^{-t}$$

$$u(x,t) = \sin x\, e^{-t} + \sin 3x\int_0^t e^{9(s-t)}\, e^{-s}\, ds + \sin 3x\int_0^t e^{8s}e^{-9t}\, ds = \sin 3x\, e^{-9t}\left(\frac{e^{8t^{-1}}}{8}\right)$$

Working rule for solving Heat equation with Non-Homogeneous boundary conditions :

$u_t = ku_{xx}$

Initial condition : $u(x,0) = f(x)$

Boundary condition : $u(0,t) = c_1$, $u(a,t) = c_2$

Let the transformation $u(x,t) = v(x,t) + Ax + B$ $\Rightarrow$ $u_t = v_t$ $\Rightarrow$ $u_{xx} = v_{xx}$

Now equation is : $V_t = kv_{xx}$,   Boundary condition : $v(0,t) = 0 = v(a,t)$

Example : $u_t = u_{xx}$, $u(0,t) = 2$, $u(1,t) = 3$, $u(x,0) = x(1-x)$

$$u(x,t) = v(x,t) + Ax + B$$

$$u(0,t) = v(0,t) + B \qquad\qquad \Rightarrow \qquad B = 2$$

$$u(1,t) = v(1,t) + A + B \qquad \Rightarrow\ A + B = 3 \ \Rightarrow\ A = 1$$

$$u(x,t) = v(x,t) + x + 2$$

$$u(x,0) = v(x,0) + x + 2 \qquad\qquad \Rightarrow \qquad v(x,0) = x - x^2 - x - 2$$

$$v(x,0) = -(x^2 + 2)$$

Now equation is : $v_t = u_{xx}$ $v(x,0) = -x^2 - 2$ $v(0,t) = v(a,t) = 0$

$$v(x,t) = \sum_{n=1}^\infty E_n \sin\frac{n\pi}{a}x\cdot e^{-k\left(\frac{n^2\pi^2}{l^2}\right)t}$$

$$E_n = \int_0^1 -(x^2 + 2)\sin n\pi x\, dx$$

Ans : $-2\left[\dfrac{1}{n\pi}(2-3\cos n\pi) + \dfrac{2}{n^3\pi^3}(\cos n\pi - 1)\right]$

Exercise 11.1

1. A rod of length 1 with insulated sides, is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0^{\circ C}$ and are kept at that temperature. Find the temperation $u(x,t)$.

2. Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $0 < x < l$, $t > 0$ given that $u(0,t) = u(l,t) = 0$ and $u(x,0) = x(l-x)$, $0 \le x \le l$.

Answers

1. $u(x,t) = \dfrac{4u_0}{\pi} \displaystyle\sum_{m=1}^{\infty} \dfrac{1}{(2m-1)} \sin\dfrac{(2m-1)\pi x}{l} e^{-c^2_{2m-1}t}$

2. $u(x,t) = \dfrac{8l^2}{\pi^3} \displaystyle\sum_{m=1}^{\infty} \dfrac{1}{(2m-1)^3} \sin\dfrac{(2m-1)\pi x}{l} e^{-\dfrac{(2m-1)^2\pi^2 t}{l^2}}$

Solution of wave equation

Solution of wave equation : Given $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$ ....(1)

Let solution of (1) be of the form $y(x,t) = X(x)T(t)$ .....(2)

From (1) and (2), $X''T = \dfrac{1}{c^2}XT'' \quad \Rightarrow \quad \dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$

since x and t are independent variables, hence the above equation can only be true if each side is equal to the same constant, say k. Thus, we obtain $X'' - kX = 0$ .....(3)

and $T'' - c^2 kT = 0$ .....(4)

Three cases arises :

Case I : When $k = 0$. Then $X = a_1 x + a_2$, $T = a_3 t + a_4$

Case II : When k is positive. Let $k = \lambda^2$ (say). Then $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$, $T = b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}$

Case III : When k is negative. Let $k = -\lambda^2$. Then $X = c_1 \cos\lambda x + c_2 \sin\lambda x$, $T = c_3 \cos c\lambda t + c_4 \sin c\lambda t$

Thus the various possible solution are $y(x,t) = (a_1 x + a_2)(a_3 t + a_4)$ .....(5)

$y(x,t) = (b_1 e^{\lambda x} + b_2 e^{-\lambda x})(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t})$ .....(6)

$y(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)(c_3 \cos c\lambda t + c_4 \sin c\lambda t)$ .....(7)

Now we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, $y(x,t)$ must be a periodic function of t. Hence $y(x,t)$ must involve trigonometric terms. Accordingly, the solution given by (7) is the only suitable solution.

Working rule for solving one-dimensional wave equation when both the ends of the string of length a are fixed and initial deflection (or shape) and velocity are prescribed :

Step I : The solution of wave equation $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$ .....(1)

Subject to the boundary conditions $y(0,t) = y(a,t) = 0$ for all t .....(2)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 9

and the initial conditions.

$$y(x,0) = f(x), \quad y_t(x,0) = g(x), \quad 0 \le x \le a \qquad \qquad .....(3)$$

is given by $y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos \dfrac{n\pi ct}{a} + F_n \sin \dfrac{n\pi ct}{a} \right) \sin \dfrac{n\pi x}{a}$ .....(4)

where

$$\left. \begin{aligned} E_n &= \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \\ \text{and} \quad F_n &= \frac{2}{n\pi c} \int_0^a g(x) \sin \frac{n\pi x}{a} dx \end{aligned} \right\} \qquad .....(5)$$

Step II : Compare the given problem with (1), (2) and (3) and find particular values of c, a, $f(x)$ and $g(x)$.

Step III : Subtitute the particular values of a, $f(x)$ and $g(x)$ in (5) and compute E_n and F_n.

Step IV : Substitute the values of E_n and F_n in (4) to arrive at the desired solution of the given boundary value problem.

Working rule for solving one-dimensional wave equation when both the ends of the string of length a are fixed and the initial velocity of the string is zero i.e., the string starts from the position of rest :

Step I : The solution of wave equation $\qquad \dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2} \dfrac{\partial^2 y}{\partial t^2}$ .....(1)

Subject to the boundary conditions $y(0,t) = y(a,t) = 0$ for all t .....(2)

the initial deflection $y(x,0) = f(x)$, $0 \le x \le a$ and initial velocity $y_t(x,0) = 0$ .....(3)

and initial velocity $\quad y_t(x,0) = 0$ .....(4)

is given by $\qquad y(x,t) = \sum_{n=1}^{\infty} E_n \cos \dfrac{n\pi ct}{a} \sin \dfrac{n\pi x}{a}$ .....(5)

where $\qquad E_n = \dfrac{2}{a} \int_0^a f(x) \sin \dfrac{n\pi x}{a} dx$ .....(6)

Step II : Compare the given boundary value problem with (1), (2), (3) and (4) and get particular values of c, a and $f(x)$. Use (5) to get E_n and then use (4) to get the required solution.

Working rule for solving one-dimensional wave equation when both the ends of the string of length *a* are fixed and the initial deflection of the string is zero :

Step I : The solution of wave equation $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$ ......(1)

Subject to the boundary conditions $y(0,t) = y(a,t) = 0$ for all t ......(2)

The initial deflection $= y(x,0) = 0, \ 0 \le x \le a$ ......(3)

and the initial velocity $= y_t(x,0) = g(x), \ 0 \le x \le a$ ......(4)

is given by $y(x,t) = \displaystyle\sum_{n=1}^{\infty} F_n \sin\dfrac{n\pi ct}{a}\sin\dfrac{n\pi x}{a}$ ......(5)

where $F_n = \dfrac{2}{x\pi c}\displaystyle\int_0^a g(x)\sin\dfrac{x\pi n}{a}dx$ ......(6)

Step II : Compare the given boundary value problem with (1), (2), (3) and (4) and compute particular values of c, a and $g(x)$.

Step III : Substitute the values of a and $g(x)$ obtained in step II in (5) to compute F_n.

Step IV : Substitute the value of F_n obtained in step III in (4) to get the required solution of the given boundary value problem.

Example 1 : Solve the one-dimensional wave equation $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}, \ 0 \le x \le 2\pi, \ t \ge 0$ .....(A)

subject to the following initial and boundary conditions

(i) $y(x,0) = \sin^3 x, \ 0 \le x \le 2\pi$ ......(B)

(ii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0, \ 0 \le x \le 2\pi$ ......(C)

(iii) $y(0,t) = y(2\pi,t) = 0, \ \text{for } t \ge 0$ ....(D)

Solution : We have to solve $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$ ....(1)

subject to the boundary conditions

$y(0,t) = y(2\pi,t) = 0$ for all t ......(2)

and the initial conditions $y(x,0) = f(x) = \sin^3 x, \ 0 \le x \le 2\pi$ ......(3)

and initial deflection $y_t(x,0) = 0, \ 0 \le x \le 2\pi$ ......(4)

Comparing the given problem given by (1), (2), (3) and (4) with the boundary value problem given by (A), (B), (C) and (D), the required solution is given by

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 11

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{nx}{2}\right)\cos\left(\frac{nct}{2}\right) \qquad \qquad ......(5)$$

Now $\quad y(x,0) = \sin^3 x$

$\Rightarrow \qquad \sin^3 x = \sum_{n=1}^{\infty} E_n \sin\left(\frac{nx}{2}\right)$

$\Rightarrow \qquad E_1 \sin\frac{x}{2} + E_2 \sin x + E_3 \sin\frac{3x}{2} + E_4 \sin 2x + E_5 \sin\frac{5x}{2} + E_6 \sin 3x + ... = \sin^3 x$

$\Rightarrow \qquad E_1 \sin\frac{x}{2} + E_2 \sin x + E_3 \sin\frac{3x}{2} + E_4 \sin 2x + E_5 \sin\frac{5x}{2} + E_6 \sin 3x + ... = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

Comparing the coefficients of the like terms on both sides, we get

$E_2 = \frac{3}{4}$, $E_6 = -\frac{1}{4}$ and $E_n = 0$ for $n \neq 2,6$

With these values of E_2, E_6 etc., (5) reduces to

$$y(x,t) = \frac{3}{4}\sin x \cos ct - \frac{1}{4}\sin 3x \cos 3ct$$

Working rule for solving Non-homogeneous wave equation :

$u_{tt} - c^2 u_{xx} = f(x,t) \qquad 0 < x < a, \quad t > 0$

Boundary condition : $u(0,t) = u(a,t) = 0$

Initial condition : $u(x,0) = \phi(x)$, $u_t(x,0) = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi}{a}ct + F_n \sin\frac{n\pi c}{a}t\right)\sin\frac{n\pi}{a}x + \sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\int_0^t f_n(s)\sin\frac{(t-s)n\pi c}{a}ds$$

Where, $f_n(t) = \frac{2}{n\pi c}\int_0^a f(x,t)\sin\frac{n\pi}{a}x\,dx$

$E_n = \frac{2}{a}\int_0^a \phi(x)\sin\frac{n\pi}{a}x\,dx \qquad\qquad F_n = \frac{2}{n\pi c}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx$

Example : $u_{tt} - u_{xx} = e^t \sin x \qquad 0 < x < \pi \qquad t > 0$

$u(0,t) = u(\pi,t) = 0 \qquad\qquad u(x,0) = \sin 2x; \quad u_t(x,0) = \sin 3x$

$$u(x,t) = \sum_{n=1}^{\infty}[E_n \cos nt + F_n \sin nt]\sin nx + \sum_{n=1}^{\infty}\sin nx\left[\int_0^t f_n(s)\sin(t-s)n\,ds\right]$$

$$E_n = \frac{2}{\pi} \int_0^\pi \sin 2x \sin nx \, dx \qquad\qquad E_2 = 1, \quad E_n = 0 \text{ for all } n - \{2\}$$

$$f_n(t) = \frac{2}{n\pi} \int_0^\pi e^t \sin x \sin nx \, dx$$

$$f_1(t) = \frac{2}{\pi} \cdot e^t \cdot \frac{\pi}{2} = e^t$$

$$\therefore \qquad u(x,t) = \cos t + 2t \sin 2x + \frac{1}{3} \sin 3t \sin 3x + \sin x \int_0^t e^t \sin(t-s) \, ds$$

Formulae : (i) $\int e^{ax} \sin(bx+c) \, dx = \dfrac{e^{ax}}{a^2 + b^2} \left[a \sin(bx+c) - b \cos(bx+c) \right]$

(ii) $\int e^{ax} \cos(bx+c) \, dx = \dfrac{e^{ax}}{a^2 + b^2} \left[a \cos(bx+c) + b \sin(bx+c) \right]$

Wave equation for infinite length : $u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty$

$$\text{With } u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

then $\quad u(x,t) = \dfrac{1}{2} \left[f(x+ct) + f(x-ct) \right] + \dfrac{1}{2c} \displaystyle\int_{x-ct}^{x+ct} g(\xi) \, d\xi$

Non-homogeneous wave equation with infinite length :

$$u_{tt} - c^2 u_{xx} = f(x,t), \qquad\qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = g(x)$$

Then $u(x,t) = \dfrac{1}{2} \left[f(x+ct) + f(x-ct) \right] + \dfrac{1}{2c} \displaystyle\int_{x-ct}^{x+ct} g(\xi) \, d\xi + \dfrac{1}{2c} \displaystyle\int_0^t \int_{x-c(t-\xi)}^{x+c(t-\xi)} f(y,\xi) \, dy \, d\xi$

Remark : Let $u(x,t)$ be the solution of wave equation $u_{tt} = c^2 u_{xx}$ and A,B,C,D are the vertices of any parallelogram. Then $u(A) + u(C) = u(B) + u(D)$.

Exercise 11.2

1. The deflection of a vibrating string of length *l*, is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$ The initial velocity is zero. The initial displacement is given by

$$y(x,0) = \begin{cases} \dfrac{x}{l}, & 0 < x < \dfrac{1}{2} \\[2mm] \dfrac{(l-x)}{l}, & \dfrac{l}{2} < x < l \end{cases}$$. Find the deflection of the string at any instant of time.

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 13

Answer

1. $y(x,t) = \dfrac{4l}{c\pi^3} \displaystyle\sum_{m=1}^{\infty} \dfrac{(-1)^{m+1}}{(2m-1)^3} \sin\dfrac{(2m-1)\pi x}{l} \sin\dfrac{(2m-1)\pi ct}{l}$

Solution of Laplace's equation

Problems based on two-dimensional Laplace's equation : The two dimensional heat equation is

given by $\dfrac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$ ......(1)

If the heat flow is steady (that is, time independent) then $\dfrac{\partial u}{\partial t} = 0$ and

(1) reduces to Laplace's equation $\nabla^2 u = 0$ i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ......(2)

A heat problem then consists of the above equation (2) to be considered in some region R of the xy-plane and a given boundary condition on the boundary curve of R. This is called a boundary value problem. We shall call it

(i) Dirichlet problem if u is prescribed on c.

(ii) Neumann problem if the normal derivative $\dfrac{\partial u}{\partial x}$ is prescribed on c.

(iii) Mixed problem if u is prescribed on a portion of c and $\dfrac{\partial u}{\partial x}$ on the remaining part of c.

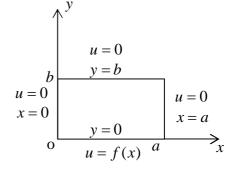The solution of (2) are called harmonic functions.

Dirichlet Problem in a rectangle : The Dirichlet problem in a rectangle is defined as follows :

Laplace's equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$

Case (i) :

Boundary conditions : $u(0,y) = 0, \ u(a,y) = 0, \ 0 \le y \le b$

$u(x,b) = 0, \quad 0 \le x \le a$

and $\qquad\qquad u(x,0) = f(x), \quad 0 \le x \le a$

Then solution is $u(x, y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)$

Where $\quad F_n = \frac{2}{a} \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\frac{n\pi x}{a} dx$

Case (ii) :

Boundary conditions : $u(x,0) = 0 \qquad u(x,b) = f(x) \qquad u(0,y) = 0 \qquad u(a,y) = 0$

Then $u(x, y) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi}{a} x \sinh\frac{n\pi}{a} y$

Where $F_n = \frac{2}{a} \frac{1}{\sinh\frac{n\pi b}{a}} \int_0^a f(x) \sin\frac{n\pi}{a} x \, dx$

Case (iii) :

Boundary conditions : $u(x,0) = 0 \qquad u(x,b) = 0 \qquad u(0,y) = g(y) \qquad u(a,y) = 0$

Then $u(x, y) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi}{b} y \sinh\frac{n\pi}{a}(a-x)$

Where $F_n = \frac{2}{b} \frac{1}{\sinh\frac{n\pi}{b} a} \int_0^b g(y) \sin\frac{n\pi}{b} y \, dy$

Case (iv) :

Boundary conditions : $u(x,0) = 0 \qquad u(x,b) = 0 \qquad u(0,y) = 0 \qquad u(a,y) = g(y)$

Then $u(x, y) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi}{b} y \sinh\frac{n\pi}{a} x$

Where $F_n = \frac{2}{b} \frac{1}{\sinh\frac{n\pi}{b} a} \int_0^b g(y) \sin\frac{n\pi}{b} y \, dy$

Example 1 : Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions $u(0, y) = u(a, y) = 0$ for $0 \le y \le b$ and $u(x,b) = 0$ and $u(x,0) = x(a-x), \ 0 \le x \le a$

Solution : For the present problem $u(x, 0) = f(x) = x(a-x) \qquad 0 \le x \le a$

Solution of given problem is given by $u(x,t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)$ ......(1)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 15

$$F_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\frac{n\pi x}{a} dx \ .$$

Now $F_n = \dfrac{2}{a \sin\left(\frac{n\pi b}{a}\right)} \displaystyle\int_0^a (ax - x^2) \sin\frac{n\pi x}{a} dx$

$$= \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \left[(ax - x^2)\left(\frac{-a}{n\pi}\right)\cos\frac{n\pi x}{a} - (a - 2x)\left(\frac{-a^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{a}\right) + (-2)\left(\frac{a^3}{n^3\pi^3}\right)\cos\frac{n\pi x}{a} \right]_0^a$$

$$= \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \left[\frac{-2a^3(-1)^n}{n^3\pi^3} + \frac{2a^3}{n^3\pi^3} \right] = \frac{4a^2}{n^3\pi^3}\left[1 - (-1)^n \right] \operatorname{cosech}\frac{n\pi b}{a}$$

$$= \begin{cases} 0, & \text{if} \quad n = 2m, \ m = 1,2,3,.... \\ \dfrac{8a^2}{\pi^3(2m-1)^3}\operatorname{cosech}\left(\dfrac{(2m-1)\pi b}{a}\right), & \text{if} \quad n = 2m-1, \ m = 1,2,3,.... \end{cases}$$

Substituting the above value of F_n in (1), the required steady temperature $u(x, y)$ is given by

$$u(x, y) = \frac{3a^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{a} \sinh\frac{(2m-1)(b-y)\pi}{a} \operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

Result : Dirichlet's problem

(i) For interior circle : $\nabla^2 u = 0$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \ \ 0 < r < a$$

With $u(a,\theta) = f(\theta)$ $0 \le \theta \le 2\pi$

Then $u(r,\theta) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$

Where $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$ $n = 0,1,2,3,.....$

$b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$ $n = 1,2,3,.....$

(ii) For exterior circle : $\nabla^2 u = 0$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a < r$$

With $u(a,\theta) = f(\theta)$ $\qquad 0 \le \theta \le 2\pi$

Then $u(r,\theta) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}\left(\dfrac{a}{r}\right)^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$

Where $a_n = \dfrac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \qquad n = 0,1,2,3,.....$

$b_n = \dfrac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta \qquad\qquad n = 1,2,3,.....$

Neumann problem for rectangle :

$u_{xx} + u_{yy} = 0 \qquad 0 \le x \le a, \qquad 0 \le y \le b$

$u_x(0,y) = u_x(a,y) = u_y(x,0) = 0, \quad u_y(x,b) = f(x)$

$u(x,y) = E_0 + \sum_{n=1}^{\infty} E_n \cosh\left(\dfrac{n\pi x}{a}\right)\cos\dfrac{n\pi y}{a}$

where $E_n = \dfrac{2}{n\pi}\dfrac{1}{\sinh\left(\dfrac{n\pi b}{a}\right)}\int_0^a x f(x)\dfrac{n\pi x}{a}\, dx$

$D_n = \dfrac{2}{n\pi}\dfrac{1}{\sinh n\dfrac{\pi b}{a}}\int_0^a f(x)\cos n\dfrac{\pi}{a}x\, dx$

Exercise 11.3

1. A rectangular plate with insulated surfaces 8 cm wide and so long compared to its width that it can be considered infinite in the length without introducing an appreciable error. If the temperature along the short edge $y = 0$ is given by $u(x,0) = 100\sin\left(\dfrac{\pi x}{8}\right)$. While the two long edges $x = 0$ and $x = 8$ as well as the other short edges are kept at 0°C. Find steady state temperature function $u(x,y)$.

Answer

1. $u(x,y) = 100\sin\left(\dfrac{\pi x}{8}\right)e^{\dfrac{-\pi y}{8}}$

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 1

---------------- **S C Q** ----------------

1. Let $u(x,t)$ be the solution of

   $u_{tt} = u_{xx}; \quad 0 < x < \infty, \quad t > 0,$

   $u(x,0) = x(1-x), u_t(x,0) = 0$.

   Then $u\left(\dfrac{1}{2},\dfrac{1}{4}\right)$ is

   1. $\dfrac{3}{16}$　　　　2. $\dfrac{1}{4}$

   3. $\dfrac{3}{4}$　　　　4. $\dfrac{1}{16}$　**(GATE 1999)**

2. The solution of the Cauchy problem

   $u_{yy}(x,y) - u_{xx}(x,y) = 0$

   $u(x,0) = 0, u_y(x,0) = x$ is $u(x,y) =$

   1. $\dfrac{x}{y}$　　　　2. xy

   3. $xy + \dfrac{x}{y}$　　　4. 0　**(GATE 2000)**

3. The solution of the initial value problem
   $u_{tt} = 4u_{xx}, \ t > 0, \ -\infty < x < \infty$. Satisfying

   the conditions $u(x,0) = x, u_t(x,0) = 0$ is

   1. x　　　　2. $\dfrac{x^2}{2}$

   3. $2x$　　　　4. $2t$　**(GATE 2001)**

4. Let u be a solution of the initial value

   problem $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0;$

   $u(x,0) = x^2, \quad \dfrac{\partial u}{\partial t}(x,0) = 0$.

   Then $u(0,1)$ equals

   1. 1　　　　2. 0

   3. 2　　　　4. $\dfrac{1}{2}$　**(GATE 2002)**

5. If $u(x,t)$ satisfies the wave equation

   $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, x \in \mathbb{R}, t > 0$ with initial

   conditions $u(x,0) = \begin{cases} \sin\dfrac{\pi x}{c}, & 0 \le x \le c \\ 0 & \text{elsewhere} \end{cases}$,

   and $u_t(x,0) = 0$ for all x, then for a given

   $t > 0$

   1. There are values of x at which $u(x,t)$ is

      discontinuous

   2. $u(x,t)$ is continuous but $u_x(x,t)$ is not

      continuous

   3. $n(x,t), u_x(x,t)$ are continuous but

      $u_{xx}(x,t)$ is not continuous

   4. $u(x,t)$ is smooth for all x

   (GATE 2004)

6. The solution of the Laplace's equation

   $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ in the unit disk

   with boundary conditions $u(1,\theta) = 2\cos^2 \theta$

   is given by

   1. $1 + r^2 \cos 2\theta$　　2. $1 + \ln r + r\cos 2\theta$

   3. $2r^3 \cos^2 \theta$　　　4. $1 - r^2 + 2r^2 \cos^2 \theta$

   (GATE 2004)

7. It is required to solve the Laplace equation

   $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, 0 < x < a, \ 0 < y < b,$

Satisfying the boundary conditions $u(x,0)=0$, $u(x,b)=0$, $u(0,y)=0$ and $u(a,y)=f(y)$. If c_n's are constants, then the equation and the homogeneous boundary conditions determine the fundamental set of solutions of the form

1. $u(x,y)=\sum_{n=1}^{\infty}c_n \sinh\dfrac{n\pi x}{b}\sin\dfrac{n\pi y}{b}$

2. $u(x,y)=\sum_{n=1}^{\infty}c_n \sin\dfrac{n\pi x}{b}\sin\dfrac{n\pi y}{b}$

3. $u(x,y)=\sum_{n=1}^{\infty}c_n \sin\dfrac{n\pi x}{b}\sinh\dfrac{n\pi y}{b}$

4. $u(x,y)=\sum_{n=1}^{\infty}c_n \sinh\dfrac{n\pi x}{b}\sinh\dfrac{n\pi y}{b}$

(GATE 2005)

8. A function $u(x,t)$, satisfies the wave equation $-\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}$, $0<x<1$, $t>0$.

If $u\left(\dfrac{1}{2},0\right)=\dfrac{1}{4}$, $u\left(1,\dfrac{1}{2}\right)=1$ and

$u\left(0,\dfrac{1}{2}\right)=\dfrac{1}{2}$ then $u\left(\dfrac{1}{2},1\right)$ is

1. $\dfrac{7}{4}$ 2. $\dfrac{5}{4}$

3. $\dfrac{4}{5}$ 4. $\dfrac{7}{4}$ **(GATE 2005)**

9. The characteristic curves for the equation $x\dfrac{\partial u}{\partial y}-y\dfrac{\partial u}{\partial x}=u$ in the (x,y) plane is

1. Straight line with slopes 1
2. Straight lines with slopes -1
3. Circles with centre at the origin
4. Circles touching y-axis and centred on x-axis   **(GATE 2006)**

10. Let PQRS be a rectangle in the first quadrant whose adjacent sides PQ and QR have slopes 1 and -1 respectively. If $u(x,t)$ is a solution of $\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial z^2}=0$ and $u(P)=1$, $u(Q)=-\dfrac{1}{2}$, $u(R)=\dfrac{1}{2}$, then $u(S)$ equals

1. 2   2. 1

3. $\dfrac{1}{2}$ 4. $-\dfrac{1}{2}$ **(GATE 2006)**

11. Let $u(x,y)$ be a solution of Laplace's equation on $x^2+y^2\le 1$. If

$u(\cos\theta,\sin\theta)=\begin{cases}\sin\theta & \text{for } 0\le\theta\le\pi \\ 0 & \text{for } \pi\le\theta\le 2\pi\end{cases}$

Then $u(0,0)$ equals

1. $\dfrac{1}{\pi}$ 2. $\dfrac{2}{\pi}$

3. $\dfrac{1}{2\pi}$ 4. $\dfrac{\pi}{2}$ **(GATE 2006)**

12. Let $u(x,t)$ be the solution of the initial value problem $\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}=0$;

$u(x,0)=\sin x, \dfrac{\partial u}{\partial t}(x,0)=1$.

Then $u(\pi,\pi/2)$ equals

1. $\dfrac{\pi}{2}$ 2. $1-\dfrac{\pi}{2}$

3. 1   4. $1+\pi$ **(GATE 2006)**

13. Let $u(x,t)$ be the solution of the one dimensional wave equation

$u_{tt}-4u_{xx}=0$, $-\infty<x<\infty$, $t>0$

$u(x,0)=\begin{cases}16-x^2, & |x|\le 4 \\ 0, & \text{otherwise}\end{cases}$ and

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   Mob. 8607383607, 9802177766

Page 3

$u_t(x,0) = \begin{cases} 1, & |x| \le 2 \\ 0, & \text{otherwise} \end{cases}$ for

$1 < t < 3, \ u(2,t) =$

1. $\dfrac{1}{2}\left[16 - (2-2t)^2\right] + \dfrac{1}{2}\left[1 - \min\{1, t-1\}\right]$

2. $\dfrac{1}{2}\left[32 - (2-2t)^2 - (2+2t)^2\right] + t$

3. $\dfrac{1}{2}\left[32 - (2-2t)^2 - (2+2t)^2\right] + 1$

4. $\dfrac{1}{2}\left[16 - (2-2t)^2\right] + \dfrac{1}{2}\left[1 - \max\{1-t, -1\}\right]$

(GATE 2007)

14. Consider the Neumann problem

$u_{xx} + u_{yy} = 0, \ 0 < x < \pi, \ -1 < y < 1$

$u_x(0,y) = u_x(\pi, y) = 0$

$u_y(x,-1) = 0, u_y(x,1) = \alpha + \beta \sin(x)$.

The problem admits solution for

1. $\alpha = 0, \ \beta = 1$ 2. $\alpha = -1, \ \beta = \dfrac{\pi}{2}$

3. $\alpha = 1, \ \beta = \dfrac{\pi}{2}$ 4. $\alpha = 1, \ \beta = -\pi$

(GATE 2007)

For question No. 15 and 16 : Consider the boundary value problem

$u_{xx} + u_{yy} = 0, \ x \in (0,\pi), \ y \in (0,\pi),$
$u(x,0) = u(x,\pi) = (0,y) = 0$

15. Any solution of this boundary value problem is of the form

1. $\displaystyle\sum_{n \in 1}^{\infty} a_n \sinh nx \sin ny$

2. $\displaystyle\sum_{n \in 1}^{\infty} a_n \cosh nx \sin ny$

3. $\displaystyle\sum_{n \in 1}^{\infty} a_n \sinh nx \cos ny$

4. $\displaystyle\sum_{n \in 1}^{\infty} a_n \cosh nx \cos ny$ **(GATE 2008)**

16. If an additional boundary condition

$u_x(\pi, y) = \sin y$ is satisfied, then

$u(x, \pi/2)$ is equal to

1. $\dfrac{\pi}{2}\left(e^{\pi} - e^{-x}\right)\left(e^{\pi} + e^{-\pi}\right)$

2. $\dfrac{\pi\left(e^x + e^{-x}\right)}{\left(e^{\pi} - e^{-\pi}\right)}$

3. $\dfrac{\pi\left(e^x - e^{-x}\right)}{\left(e^{\pi} + e^{-\pi}\right)}$

4. $\dfrac{\pi}{2}\left(e^{\pi} + e^{-\pi}\right)\left(e^x + e^{-x}\right)$ **(GATE 2008)**

17. Let $u(x,t)$ be the solution of

$u_{tt} - u_{xx} = 1, \ x \in R, \ t > 0,$ with

$u(x,0) = 0, \ u_t(x,0) = 0, \ x \in \mathbb{R}$ then

$u(1/2, 1/2)$

1. $\dfrac{1}{8}$ 2. $-\dfrac{1}{8}$

3. $\dfrac{1}{4}$ 4. $-\dfrac{1}{4}$ **(GATE 2008)**

18. For the diffusion problem

$u_{xx} = u_t \ (0 < x < \pi, \ t > 0), \ u(0,t) = 0,$

$u(\pi, t) = 0$ and $u(x,0) = 3\sin 2x$ the

solution is given by

1. $3e^{-t} \sin 2x$ 2. $3e^{-4t} \sin 2x$

3. $3e^{-9t}\sin 2x$ 4. $3e^{-2t}\sin 2x$

(GATE 2009)

19. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ t > 0\ \text{with}$$

$u(0,t) = u(\pi,t) = 0,\ u(x,0) = \sin x$ and

$\dfrac{\partial u}{\partial t} = 0$ at $t = 0$. Then $u\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is

1. 2   2. 1
3. 0   4. -1 **(GATE 2010)**

20. The vertical displacement $u(x,t)$ of an infinitely long elastic string is governed by the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ t > 0,\ u(x,0) = -x$$

and $\dfrac{\partial u}{\partial t}(x,0) = 0$. The value of $u(x,t)$ at

$x = 2$ and $t = 2$ is equal to

1. 2   2. 4
3. -2 4. -4 **(GATE 2011)**

21. The diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\quad u = u(x,t),\ u(0,t) = 0 = u(\pi,t),$$
$u(x,0) = \cos x \sin 5x$, admits the solution

1. $\dfrac{e^{-36t}}{2}\left[\sin 6x + e^{20t}\sin 4x\right]$

2. $\dfrac{e^{-36t}}{2}\left[\sin 4x + \dfrac{e^{20t}}{2}\left\{\sin 4x + e^{20t}\sin 6x\right\}\right]$

3. $\dfrac{e^{-20t}}{2}\left[\sin 3x + e^{15t}\sin 5x\right]$

4. $\dfrac{e^{-36t}}{2}\left[\sin 5x + e^{20t}\sin x\right]$ **(GATE 2012)**

22. Let $u(x,t)$ be the solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t),\ u(x,0) = \cos(5\pi x),$$

$\dfrac{\partial u}{\partial t}(x,0) = 0$. Then, the value of $u(1,1)$ is

1. 1   2. 3
3. -1 4. -3 **(GATE 2013)**

23. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\qquad t > 0,\ \text{with the}$$

boundary conditions

$u(0,t) = 0,\ u(\pi,t) = 0$ for $t > 0$ and the

initial condition $u(x,0) = \sin x$. Then

$u\left(\dfrac{\pi}{2}, 1\right)$ is

1. $\dfrac{\pi}{e}$ 2. $\dfrac{2\pi}{e}$

3. $\dfrac{\pi}{2e}$ 4. None   **(GATE 2014)**

24. If $u(x,t)$ is the D'Alembert's solution to

the wave equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2},\ x \in \mathbb{R},\ t > 0$,

with the condition $u(x,0) = 0$ and

$\dfrac{\partial u}{\partial t}(x,0) = \cos x$, then $u\left(0, \dfrac{\pi}{4}\right)$ is

1. 4   2. $\dfrac{1}{2}$

3. $\dfrac{1}{3}$ 4. $\dfrac{1}{\sqrt{2}}$ **(GATE 2014)**

25. Let $u(x,t)$ be the d'Alembert's solution of the initial value problem for the wave equation $u_{tt} - c^2 u_{xx} = 0$

$u(x,0) = f(x),\ u_t(x,0) = g(x)$, where c is

a positive real number and f, g are smooth

odd functions. Then $u(0,1)$ is

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com **Mob. 8607383607, 9802177766**

Page 5

1. 4   2. 3

3. 0   4. 2   **(GATE 2014)**

26. Let $u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, be the solution of

the initial value problem

$u_{tt} = u_{xx}$, $u(x,0) = x\, u_t(x,0) = 1$.

Then $u(2,2)$

1. 1   2. 3   3. 4   4. 2

         (GATE 2015)

27. Let $\Omega = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < 1 \right\}$ be the

open unit disc in $\mathbb{R}^2$ with boundary $\partial\Omega$. If

$u(x,y)$ is the solution of the Dirichlet

problem $u_{xx} + u_{yy} = 0$ in Ω

$u(x,y) = 1 - 2y^2$ on $\partial\Omega$, then $u\left(\dfrac{1}{2},0\right)$ is

equal to

1. -1 2. $-\dfrac{1}{4}$

3. $\dfrac{1}{4}$ 4. 1   **(GATE 2015)**

28. Let $u(r,\theta)$ be the bounded solution of the

following boundary value problem in polar

coordinates :

$r^2 \dfrac{\partial^2 u}{\partial r^2} + r\dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$, $0 < r < 2$ and

$0 \leq \theta \leq 2\pi$, $u(2,\theta) = \cos^2\theta$, $0 \leq \theta \leq 2\pi$.

Then $u(1, \pi/2) + u(1, \pi/4)$ equals

1. 1   2. $\dfrac{9}{8}$

3. $\dfrac{7}{8}$ 4. $\dfrac{3}{8}$ **(GATE 2017)**

29. The function

$$u(x,t) = \begin{cases} \dfrac{1}{\sqrt{t}} e^{\frac{-x^2}{4t}}, & t > 0,\ x \in \mathbb{R} \\ 0, & t \leq 0,\ x \in \mathbb{R} \end{cases} \text{ is a}$$

solution of the heat equation in

1. $\left\{ (x,t) : x \in \mathbb{R}, t \in \mathbb{R} \right\}$

2. $\left\{ (x,t) : x \in \mathbb{R}, t > 0 \right\}$ but not in the set

   $\left\{ (x,t) : x \in \mathbb{R}, t < 0 \right\}$

3. $\left\{ (x,t) : x \in \mathbb{R}, t \in \mathbb{R} \right\} \setminus \left\{ (0,0) \right\}$

4. $\left\{ (x,t) : x \in \mathbb{R}, t < -1 \right\}$

      (CSIR NET SCQ June 2012)

30. Let $u(x,t)$ be the solution of the initial

value problem

$u_{tt} - u_{xx} = 0$

$u(x,0) = x^3$

$u_t(x,0) = \sin x$

Then $u(\pi,\pi)$ is

1. $4\pi^3$ 2. π^3

3. 0   4. 4

      (CSIR NET Dec 2017)

31. Let u be the unique solution of

$$\left. \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0,\ x \in \mathbb{R},\ t > 0 \\[2mm] u(x,0) = f(x),\ \dfrac{\partial u}{\partial t}(x,0) = 0,\ x \in \mathbb{R} \end{array} \right\}$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies the relations

$f(x) = x(1-x)$ $\forall$ $x \in [0,1]$ and

$f(x+1) = f(x)$ $\forall$ $x \in \mathbb{R}$.

Then $u\left(\dfrac{1}{2}, \dfrac{5}{4}\right)$ is

1. $\dfrac{1}{8}$ 2. $\dfrac{1}{16}$

3. $\dfrac{3}{16}$ 4. $\dfrac{5}{16}$

(CSIR NET June 2018)

--------------------- M C Q -------------------

$$\left.\begin{array}{l} u_{xx} + u_{yy} + \lambda u = 0, \ 0 < x, y < 1 \\ u(x,0) = u(x,1) = 0, \ 0 \le x \le 1 \\ u(0,y) = u(1,y) = 0, \ 0 \le y \le 1 \end{array}\right\}$$

1. The PDE

   has

   1. A unique solution u for any $\lambda \in \mathbb{R}$
   2. Infinitely many solutions for some

      $\lambda \in \mathbb{R}$
   3. A solution for countably many values of λ
   4. Infinitely many solutions for all $\lambda \in \mathbb{R}$

      (CSIR NET MCQ June 2011)

2. Let $u(x,t)$ be the solution of the initial

   boundary value problem

   $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < \infty, \ t > 0 \,,$$

   $$u(x,0) = \cos\left(\frac{\pi x}{2}\right), \ 0 \le x < \infty,$$

   $$\frac{\partial u}{\partial t}(x,0) = 0, \ 0 \le x < \infty, \ \frac{\partial u}{\partial x}(0,t) = 0, \quad t \ge 0$$

   1. The value of $u(2,2) = -1$
   2. The value of $u(2,2) = 1$
   3. The value of $u\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = \dfrac{1}{\sqrt{2}}$
   4. The value of $u\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = \dfrac{1}{2}$

   (CSIR NET MCQ Dec 2011)

3. Let u be a solution of the heat equation

$$\left.\begin{array}{l} u_t - u_{xx} = 0 \quad 0 < x < \pi \text{ and } t > 0 \\ u(0,t) = u(\pi,t) = 0 \qquad t > 0 \\ u(x,0) = \sin x + \sin 2x, \quad 0 \le x \le \pi \end{array}\right\}.$$

Then

1. $u(x,t) \to 0$ as $t \to \infty$ for all $x \in (0,\pi)$
2. $t^2 u(x,t) \to 0$ as $t \to \infty$ for all $xc(0,\pi)$
3. $e^2 u(x,t)$ is a bounded function for

   $x \in (0,\pi), \ t > 0$
4. $e^{2t} u(x,t) \to 0$ as $t \to \infty$ for all

   $x \in (0,\pi)$

   (CSIR NET MCQ June 2012)

4. Consider the Laplace equation in polar form

   :

   $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0; \ 0 < r \le a, \ 0 \le \theta < 2\pi$$

   satisfying $u(a,\theta) = f(\theta)$, where f is a

   given function. Let σ be the separation

   constant that appears when one uses the

   method of separation of variables. Then for

   solution $u(r,\theta)$ to be bounded and also

   periodic in θ with period 2π .

   1. σ cannot be negative
   2. σ can be zero, and in that case the

      solution is a constant
   3. σ can be be positive and in that case it

      must be an integer
   4. the fundamental set of solutions

      $\{1, r^n \sin n\theta, r^n \cos n\theta\}$, where n is a

      positive integer

      (CSIR NET MCQ June 2013)

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 7

5. Let the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}, t \geq 0, \overline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$$

admits an exponential function $e^{-\left(i\left(\vec{k}\cdot\vec{x}+wt\right)\right)}$

as its solution, where $\vec{k}$ a nonzero constant real vector, and w is a constant. Then the solution

1. remains constant on certain planes in $\mathbb{R}^3$

2. repeats itself after a certain length L

3. has, in general, an amplitude decaying exponentially with time t

4. is bounded uniformly for $\vec{x} \in \mathbb{R}^3$ for a fixed t

(CSIR NET MCQ June 2013)

6. Let $u(x,t)$ be the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$ which tends to zero as $t \to \infty$

and has the value $\cos(x)$ when $t = 0$ then

1. $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n) e^{-nt}$ where a_n, b_n

    are arbitrary constants.

2. $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n) e^{-n^2 t}$ where a_n, b_n

    are non-zero constants.

3. $u = \sum_{n=1}^{\infty} a_n \cos(nx + b_n) e^{-nt}$ where a_n are

    not all zero and $b_n = 0$ for $n \geq 1$.

4. $u = \sum_{n=1}^{\infty} a_n \cos(nx + b_n) e^{-n^2 t}$ where

    $a_1 \neq 0$, $a_n = 0$ for $n > 1$ and $b_n = 0$

for $n \geq 1$.

(CSIR NET MCQ June 2014)

7. Let $u(x, y)$ be the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$, which tends to zero as

$y \to \infty$ and has the value $\sin x$ when

$y = 0$. Then

1. $u = \sum_{n=1}^{x} a_n \sin(nx + b_n) e^{-ny}$; where a_n are

    arbitrary and b_n are non-zero constants.

2. $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n) e^{-n^2 y}$; where

    $a_n = 1$ and $a_n(n > 1)$, b_n are non-zero

constants.

3. $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n) e^{-ny}$; where

    $a_1 = 1$, $a_n = 0$ for $n > 1$ and $b_n = 0$ for

    $n \geq 1$.

4. $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n) e^{-n^2 y}$; where

    $b_n = 0$ for $n \geq 0$ and a_n are all

nonzero.

(CSIR NET MCQ Dec 2015)

8. Let $u(x,t)$ satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}; \, x \in (0, 2\pi), t > 0$$

$u(x,0) = e^{i\omega x}$ for some $\omega \in \mathbb{R}$. Then

1. $u(x,t) = e^{i\omega x} e^{i\omega t}$

2. $u(x,t) = e^{i\omega x} e^{-i\omega t}$

3. $u(x,t) = e^{i\omega x}\left(\dfrac{e^{i\omega t}+e^{-i\omega t}}{2}\right)$

4. $u(x,t) = t + \dfrac{x^2}{2}$

9. Let u be the solution of the boundary value problem $u_{xx} + u_{yy} = 0$ for $0 < x, y < \pi$

$u(x,0) = 0 = u(x,\pi)$

$u(0,y) = 0, \quad u(\pi, y) = \sin y + \sin 2y$

for $0 \le x \le \pi$. Then

1. $u\left(1,\dfrac{\pi}{2}\right) = \left(\sinh(\pi)\right)^{-1}\sinh(1)$

2. $u\left(1,\dfrac{\pi}{2}\right) = \left(\sinh(1)\right)^{-1}\sinh(\pi)$

3. $u\left(1,\dfrac{\pi}{4}\right) = \left(\sinh(\pi)\right)^{-1}\sinh(1)\dfrac{1}{\sqrt{2}}$

   $+\left(\sinh(2\pi)^{-1}\sinh(2)\right)$

4. $u\left(1,\dfrac{\pi}{4}\right) = \left(\sinh(1)\right)^{-1}\sinh(\pi)\dfrac{1}{\sqrt{2}}$

   $+\left(\sinh(2)^{-1}\sinh(2\pi)\right)$

10. Let $u : \mathbb{R} \in [0,\infty) \to \mathbb{R}$ be a solution of the initial value problem

$\left.\begin{array}{c} u_{tt} - u_{xx} = 0, \text{ for } (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x), x \in \mathbb{R} \\ u_t(x,0) = g(x), x \in \mathbb{R} \end{array}\right\}$

Suppose $f(x) = g(x) = 0$ for $x \notin [0,1]$, then we always have

1. $u(x,t) = 0$ for all

   $(x,t) \in (-\infty,0) \times (0,\infty)$

2. $u(x,t) = 0$ for all $(x,t) \in (1,\infty) \times (0,\infty)$

3. $u(x,t) = 0$ for all (x,t) satisfying

   $x + t < 0$

4. $u(x,t) = 0$ for all (x,t) satisfying

   $x - t > 1$

11. Let $u : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be a C^2 function

satisfying $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, for all

$(x,y) \ne (0,0)$. Suppose u is of the form

$u(x,y) = f\left(\sqrt{x^2 + y^2}\right)$ where

$f : (0,\infty) \to \mathbb{R}$, is a non constant function,

then

1. $\lim\limits_{x^2+y^2 \to 0} |u(x,y)| = \infty$

2. $\lim\limits_{x^2+y^2 \to 0} |u(x,y)| = 0$

3. $\lim\limits_{x^2+y^2 \to \infty} |u(x,y)| = \infty$

4. $\lim\limits_{x^2+y^2 \to \infty} |u(x,y)| = 0$

12. Consider the wave equation for $u(x,t)$

$\left.\begin{array}{c} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0, (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x), x \in \mathbb{R} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), x \in \mathbb{R} \end{array}\right\}$

Let u_i be the solution of the above

problem with $f = f_i$ and $g = g_i$ for $i = 1, 2$

where $f_i : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R} \to \mathbb{R}$ are

given C^2 functions satisfying

$f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, for

J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
E-mail us on - jrinstituterohtak@gmail.com, balwanmudgil54@gmail.com   **Mob. 8607383607, 9802177766**

Page 9

every $x \in [-1,1]$. Which of the following

statements are necessarily true ?

1. $u_1(0,1) = u_2(0,1)$

2. $u_1(1,1) = u_2(1,1)$

3. $u_1\left(\frac{1}{2},\frac{1}{2}\right) = u_2\left(\frac{1}{2},\frac{1}{2}\right)$

4. $u_1(0,2) = u_2(0,2)$

(CSIR NET MCQ Dec 2016)

13. Let B be the unit ball in $\mathbb{R}^2$. Let

$u \in C^2(\bar{B})$ be a minimizer of

$I(u) = \int_B \left(|\nabla u|^2 + fu\right)dx + \int_{\partial B} au^2\, ds$ where f

and a are continuous functions in $C^2(\bar{B})$.

Let $\bar{n}$ denote the unit outward normal.

Which of the following are correct ?

1. $-2\Delta u + f = 0$ in B and $\frac{\partial u}{\partial n} + au = 0$ on ∂B

2. $-2\Delta u + f + a = 0$ in B and $\frac{\partial u}{\partial \bar{n}} + au = 0$

    on ∂B

3. $-\Delta u + f = 0$ in B and $2\frac{\partial u}{\partial \bar{n}} + au = 0$ on

    ∂B

4. $-\Delta u + 2f = 0$ in B and $2\frac{\partial u}{\partial \bar{n}} + au = 0$ on

    ∂B **(CSIR NET MCQ June 2017)**

14. If $u(x,t)$ is the solution of

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\, 0 < x < 1, t > 0$

$u(x,0) = 1 + x + \sin(\pi x)\cos(\pi x)$

$u(0,t) = 1,\, u(1,t) = 2$

then

1. $u\left(\frac{1}{2},\frac{1}{4}\right) = \frac{3}{2}$

2. $u\left(\frac{1}{2},\frac{1}{2}\right) = \frac{3}{2}$

3. $u\left(\frac{1}{4},\frac{3}{4}\right) = \frac{5}{4} + \frac{1}{2}e^{-3\pi^2}$

4. $u\left(\frac{1}{4},1\right) = \frac{5}{4} + \frac{1}{2}e^{-4\pi^2}$

(CSIR NET June 2018)

Answer Key

SCQ

1. 1	2. 2	3. 1
4. 1	5. 4	6. 1
7. 1	8. 2	9. 3
10. 1	11. 1	12. 1
13. 2	14. 2	15. 1
16. 3	17. 1	18. 2
19. 4	20. 4	21. 1
22. 1	23. 4	24. 4
25. 3	26. 3	27. 3
28. 3	29. 3	30. 1
31. 3		

MCQ

1. 2,3	2. 2,3	3. 1,2,3
4. 1,2,3,4	5. 1,2,3,4	6. 3,4
7. 3	8. 1,2,3	9. 1,2
10. 3,4	11. 1,3	12. 1,3
13. 1	14. 1,2,3,4	